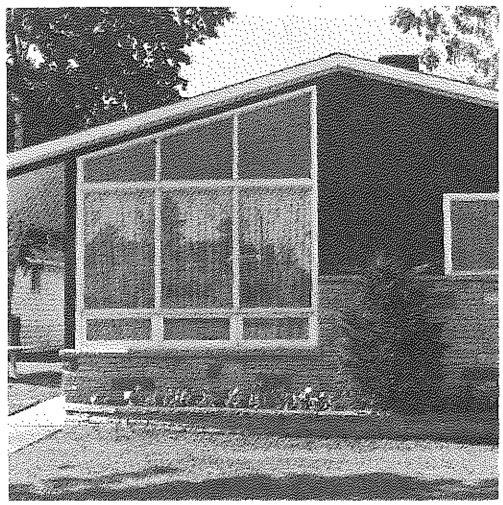
9 Geometry

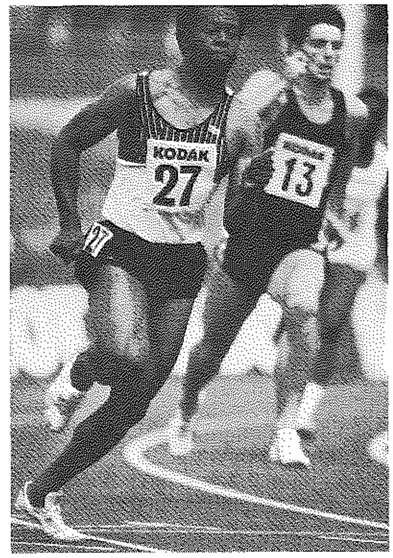
Sami and Jane have a measuring tape and a compass. How can they measure the width of a river they cannot cross? (See Section 9-5, *Example 2*.)

9-1 WHAT IS GEOMETRY?

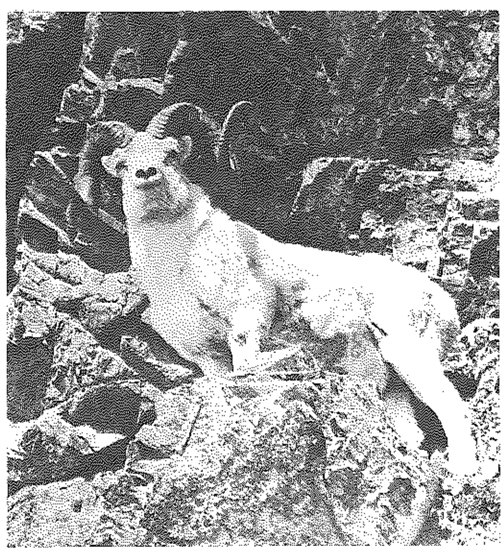
Geometry is all around us, as the photographs on these pages show. Study each photograph and the question that accompanies it. The questions and your answers involve some of the ideas and the language of geometry.



Architecture
What shapes do you see?



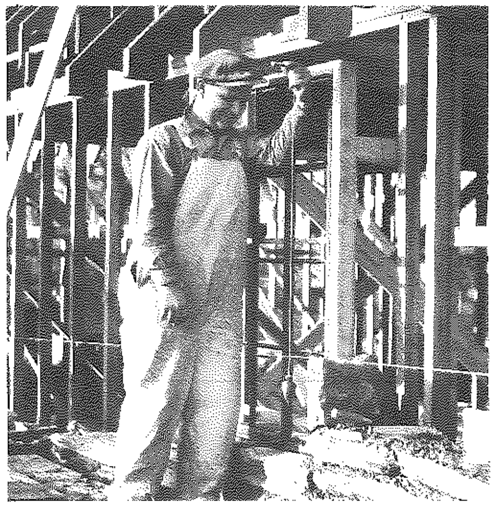
Sports
Why do runners on an oval track start at different places?



Nature
What is the shape of the horns?

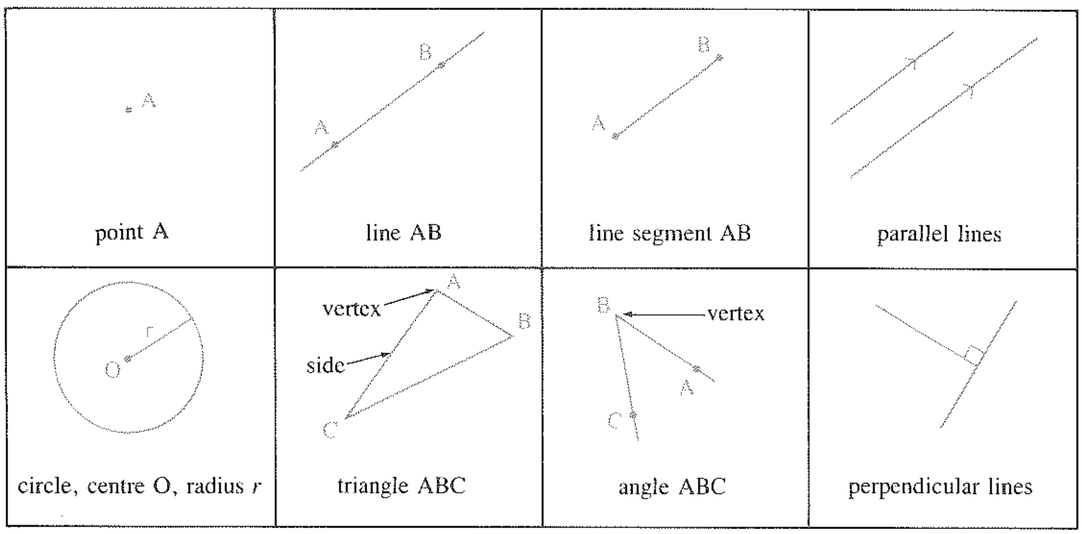
Navigation
What measurements does a navigator make?

Art
How is the feeling of "depth" achieved?



Building and Decorating
What is the purpose of the weighted string?

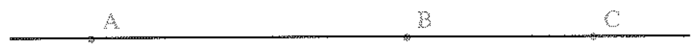
The answers to these questions required knowledge of the following basic geometric concepts.



The study of these concepts is the basis of geometry. The ancient Egyptians and Babylonians used geometric ideas to determine the areas of fields and the volumes of buildings such as temples and pyramids.

The word *geometry* comes from two Greek words *geos* and *metron* meaning earth measure. It was about 300 B.C. when the Greeks began to study geometry. Their contributions have influenced the study of the subject to the present time.

Example 1. Name all the line segments in this line.

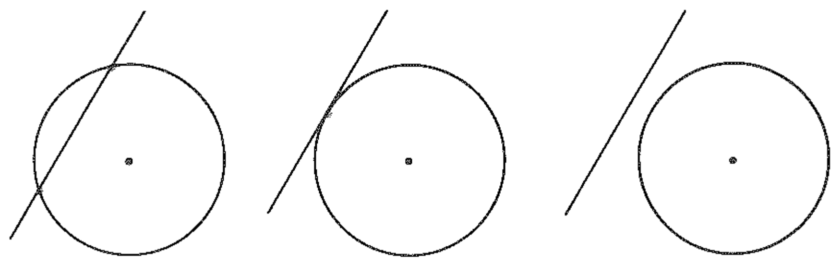


Solution. The line segments are AB, BC, and AC.

Problems in geometry often involve investigations of relationships among the basic geometric concepts.

Example 2. Find in how many points a line and a circle can intersect.

Solution. Draw a diagram to show each case.



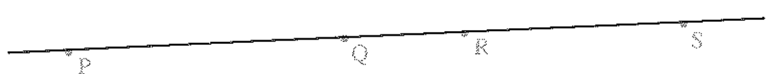
two points of intersection — one point of intersection — no points of intersection

A line and a circle can intersect in 2 points, in 1 point, or not at all.

EXERCISES 9-1

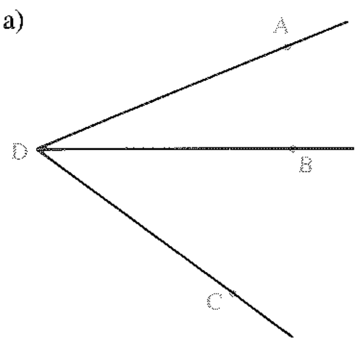
(A)

1. Collect some pictures that show geometry in the world around us.
2. Name all the line segments in this line.

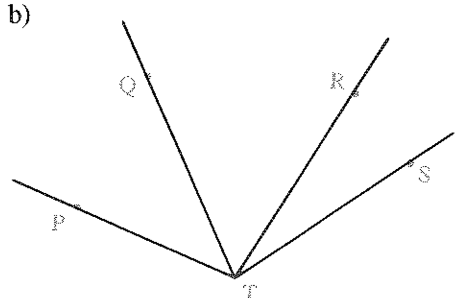


3. Name all the angles in each figure.

a)

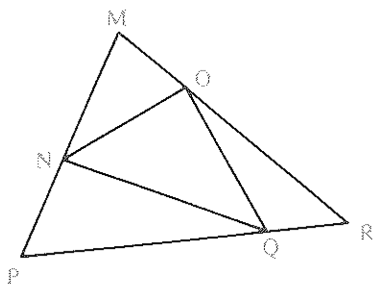


b)

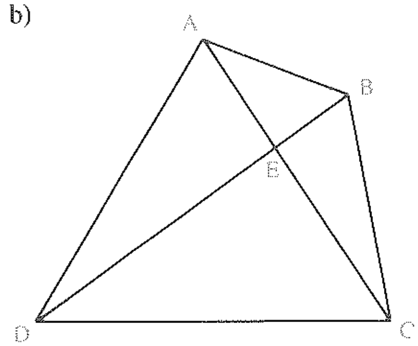


4. Name all the triangles in each figure.

a)



b)

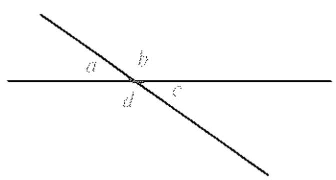


(B)

5. Find in how many points two circles can intersect if:
   a) the circles have equal radii b) the circles have different radii.
6. Find in how many points two lines can intersect.
7. Find in how many points these figures can intersect.
   a) a line segment and a line b) a line and a triangle
8. Find the greatest number of points in which each pair of figures can intersect.
   a) a triangle and a circle b) two triangles c) a square and a circle

INVESTIGATE

Two intersecting lines form two pairs of angles called *opposite angles*.

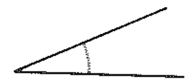
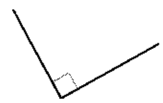
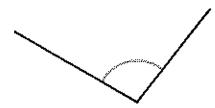
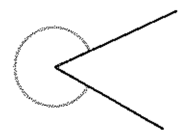


a and c are opposite angles.
b and d are opposite angles.

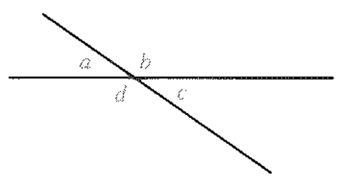
1. Draw two intersecting lines. Label and measure both pairs of opposite angles.
2. Repeat the procedure for another pair of intersecting lines.
3. Do pairs of opposite angles appear to have a special property?
4. Write a statement to describe a property of a pair of opposite angles formed when two lines intersect.

9-2 ANGLES AND INTERSECTING LINES

Angles are classified according to their measures in degrees.
One degree (1°) is $\frac{1}{360}$ of a complete rotation.

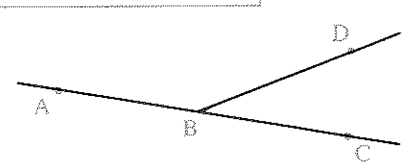
Measure	Angle	Example
Less than 90°	acute	
90°	right	
Between 90° and 180°	obtuse	
180°	straight	
Between 180° and 360°	reflex	

From the previous *INVESTIGATE*, you may have discovered that, in this situation, $a = c$ and $b = d$.

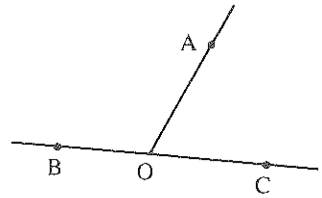


When two lines intersect, the opposite angles are equal.

Two angles with a sum of 180° are *supplementary angles*.
ABC is a straight line.
$\angle ABD + \angle DBC = 180°$
$\angle ABD$ and $\angle DBC$ are supplementary.



Example 1. ∠BOA and ∠AOC are supplementary angles.
∠AOC = 65°
Find the measure of ∠BOA.

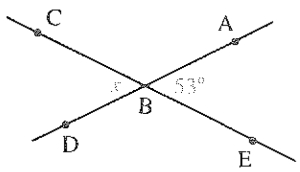


Solution. Since ∠BOA and ∠AOC are supplementary,

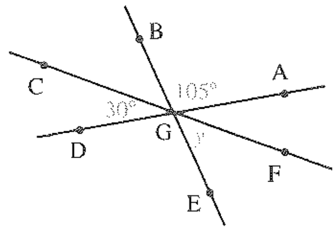
∠BOA + ∠AOC = 180°
∠BOA + 65° = 180°
∠BOA = 115°

Example 2. Find the angle measure indicated by each letter.

a)



b)



Solution. a) Since ∠CBD and ∠EBA are opposite angles,

∠CBD = ∠EBA
x = 53°

b) Since ∠DGE and ∠AGB are opposite angles,

∠DGE = ∠AGB
∠DGE = 105°

Since ∠CGF is a straight angle,

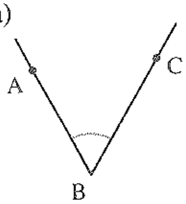
∠CGD + ∠DGE + ∠EGF = 180°
30° + 105° + y = 180°
135° + y = 180°
y = 45°

EXERCISES 9-2

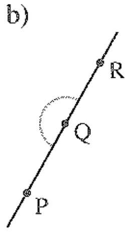
(A)

1. Identify each angle as acute, right, obtuse, straight, or reflex.

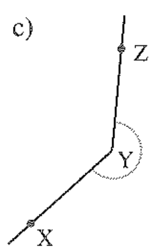
a)



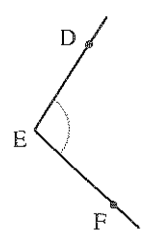
b)



c)

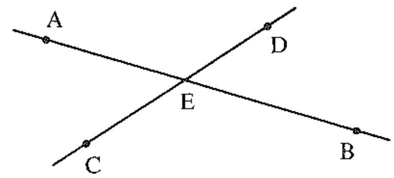


d)

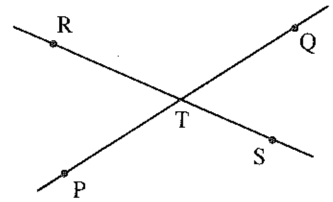


2. Name two pairs of opposite angles in each figure.

a)

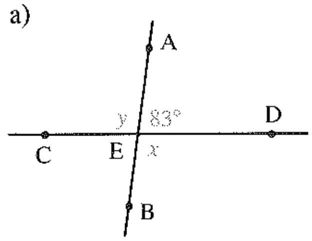


b)

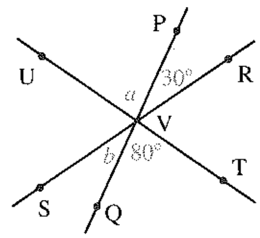


3. Find the angle measure indicated by each letter.

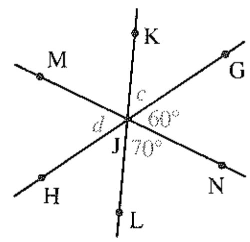
a)



b)



c)



B

4. Plot these points on a grid: P(8,2), Q(0, −2), R(2,4), and S(6, −4). Draw line segments PQ and RS. Measure the opposite angles formed by PQ and RS.

5. Plot each set of points on a grid. Draw each △ABC. Identify each angle.
   a) A(4,0), B(−3, −2), C(0,5)
   b) A(7, −1), B(2,2), C(−5,0)
   c) A(−4,8), B(−1,2), C(7,6)

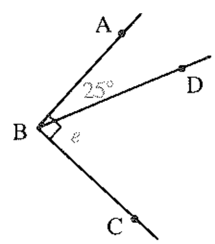
6. Two lines intersect to form four equal angles. What can be said about the lines?

7. Find in how many points a line and two parallel lines can intersect. Consider all possible cases.

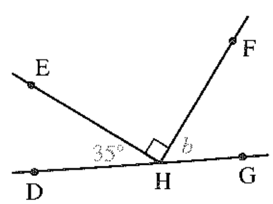
8. Angle ABC has a measure of 74°. What is the measure of reflex angle ABC?

9. Find the value of each letter.

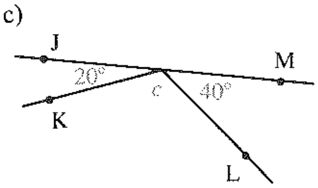
a)



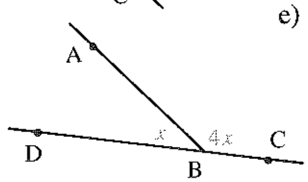
b)



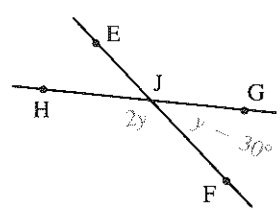
c)



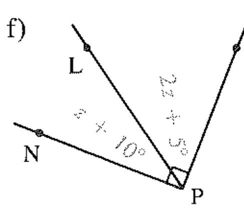
d)



e)



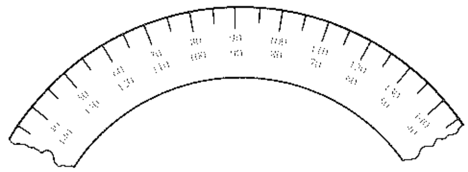
f)



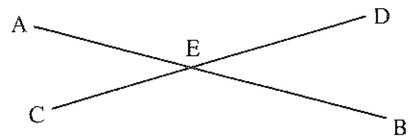
Ⓒ

10. Find in how many points these figures can intersect. Consider all possible cases.
    a) two parallel lines and a circle
    b) two parallel lines and a line segment

11. The adjacent numbers on the two scales of a double-scale protractor add up to 180°. Explain why.



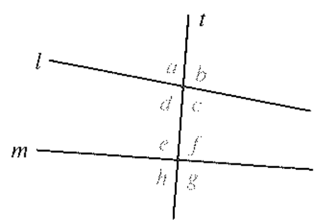
12. a) Two line segments AB and CD intersect at point E. Explain why each statement is true.
    i) $\angle AED + \angle DEB = 180°$
    ii) $\angle AED + \angle AEC = 180°$
    b) Use the equations in part a) to conclude that $\angle DEB = \angle AEC$.



INVESTIGATE

A line that intersects two or more lines is called a *transversal*. When a transversal intersects two other lines, the pairs of angles formed are described by their relative positions.

Transversal t intersects lines l and m.



There are two pairs of *alternate angles*: d and f; c and e.
There are four pairs of *corresponding angles*: a and e; b and f; d and h; c and g.
There are two pairs of *interior angles*: d and e; c and f.

When a transversal intersects two parallel lines, each pair of angles described above has a special property.

1. Draw two parallel lines and a transversal. Label and measure:
   a) each pair of alternate angles
   b) each pair of corresponding angles
   c) each pair of interior angles.
2. Repeat the procedure for another pair of parallel lines.
3. What property does each pair of angles appear to have?
4. Write a statement to describe for parallel lines a property of:
   a) a pair of alternate angles
   b) a pair of corresponding angles
   c) a pair of interior angles.

Do these properties hold for non-parallel lines?

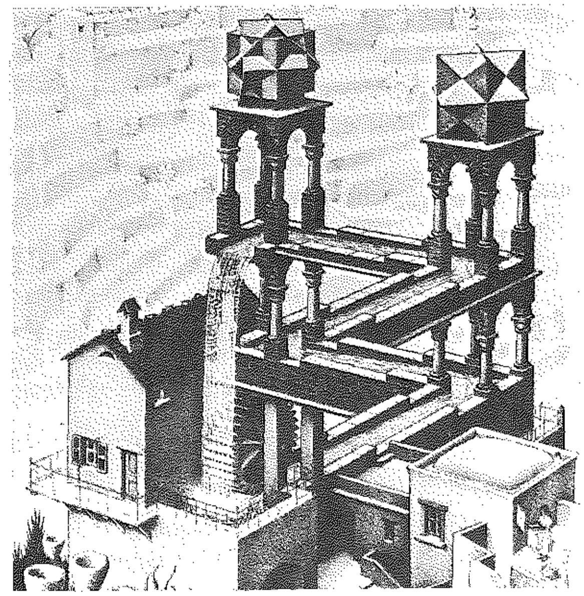
THE MATHEMATICAL MIND

Misleading Diagrams

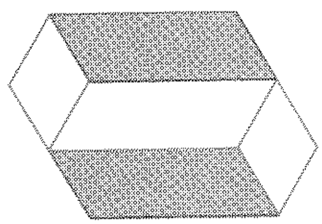
Maurits Cornelis Escher was an artist, who was born in the Netherlands in 1898. He travelled extensively through Europe before his death in 1972.

Much of Escher's work is unique; one of his many talents was the ability to design and draw "impossible" pictures. What is impossible in the illustration of the waterfall?

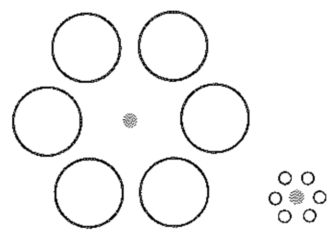
When geometric figures are combined, the resulting diagrams can sometimes deceive the eye. This can happen in different ways.



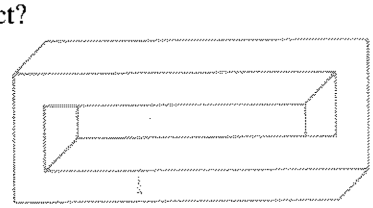
Reversing Diagrams
Some diagrams can be seen in different ways. What do *you* see?



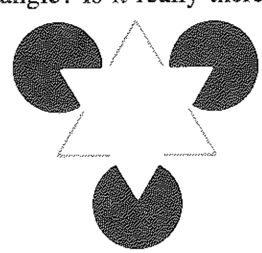
Optical Illusions
An optical illusion can lead to a false conclusion. Are the colored circles the same size?



Impossible Objects
A two-dimensional diagram can be drawn of a three-dimensional object that cannot exist. Do you think you could make this object?



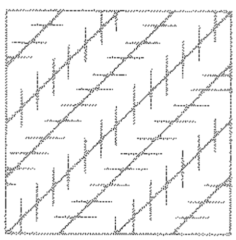
Subjective Contours
Sometimes the outline of a figure is visible when it is not really there. Do you see a white triangle? Is it really there?



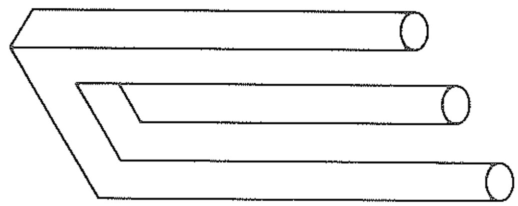
These examples show how careful we must be when drawing conclusions from a diagram.

QUESTIONS

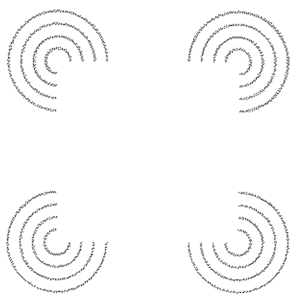
1. Are the diagonal line segments parallel?



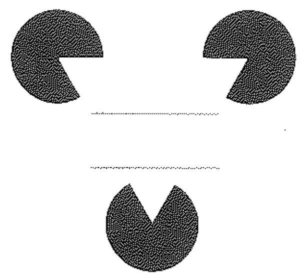
2. Can you make a physical model of this drawing?



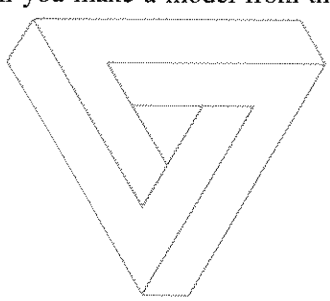
3. Is there really a white square?



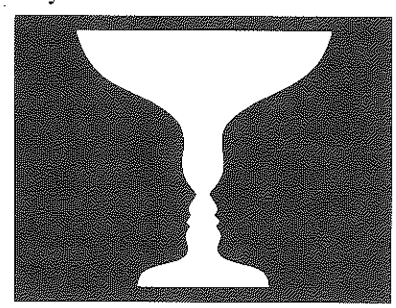
4. Do you see a white triangle? Are the line segments equal in length?



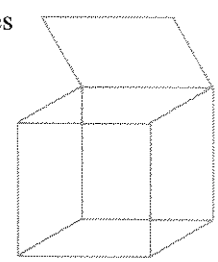
5. Can you make a model from this diagram?



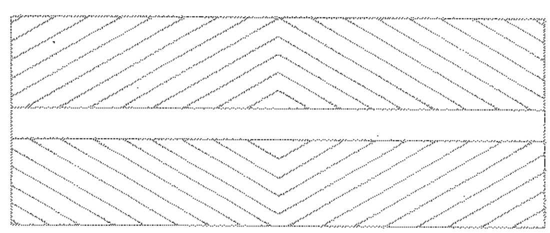
6. Do you see two heads or a birdbath?



7. How many ways does this box open?



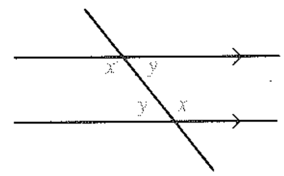
8. Are the horizontal lines "bent"?



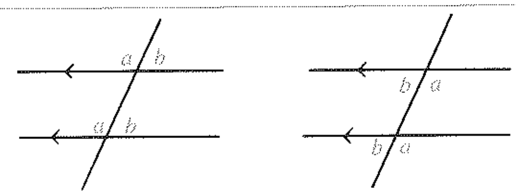
9-3 ANGLES AND PARALLEL LINES

Two lines, in the same plane, that never meet are called *parallel lines*.
From the previous *INVESTIGATE*, you may have discovered the following properties of parallel lines.

When a transversal intersects two parallel lines, the alternate angles are equal.

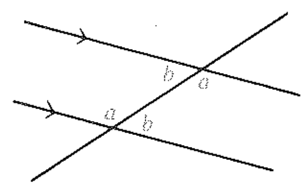


When a transversal intersects two parallel lines, the corresponding angles are equal.



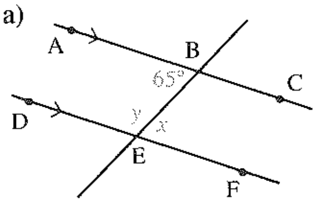
When a transversal intersects two parallel lines, the interior angles are supplementary.

$a + b = 180°$

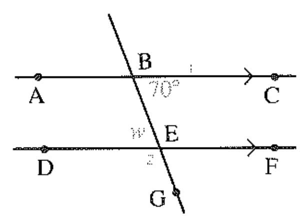


Example 1. Find the angle measure indicated by each letter.

a)



b)



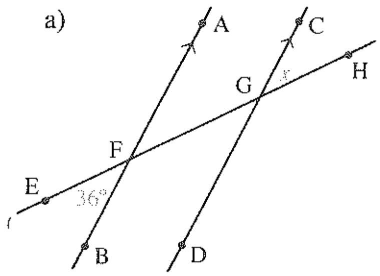
Solution.

a) Since ∠FEB and ∠ABE are alternate angles between parallel lines,
∠FEB = ∠ABE
$x = 65°$
Since ∠DEF is a straight angle,
$y + x = 180°$
$y + 65° = 180°$
$y = 115°$

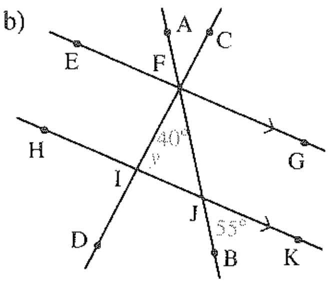
b) Since ∠DEB and ∠CBE are alternate angles between parallel lines,
∠DEB = ∠CBE
$w = 70°$
Since ∠GEB is a straight angle,
$w + z = 180°$
$70° + z = 180°$
$z = 110°$

Example 2. Find the angle measure indicated by each letter.

a)



b)



Solution. a) Since ∠FGD and ∠EFB are corresponding angles between parallel lines,

∠FGD = ∠EFB
= 36°

Since ∠CGH and ∠FGD are opposite angles,

∠CGH = ∠FGD
x = 36°

b) Since ∠GFJ and ∠KJB are corresponding angles between parallel lines,

∠GFJ = ∠KJB
= 55°

∠GFI = ∠GFJ + ∠JFI
= 55° + 40°
= 95°

Since ∠GFI and ∠FIJ are interior angles between parallel lines,

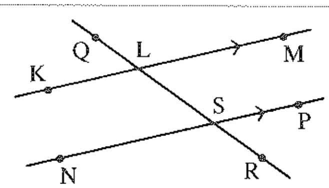
∠GFI + ∠FIJ = 180°
95° + y = 180°
y = 85°

In this example, can you suggest other ways of finding the values of x and y?

EXERCISES 9-3

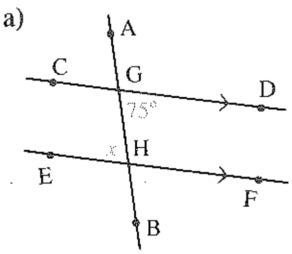
Ⓐ

1. In this figure
   a) Name two pairs of alternate angles.
   b) Name four pairs of corresponding angles.
   c) Name two pairs of interior angles.

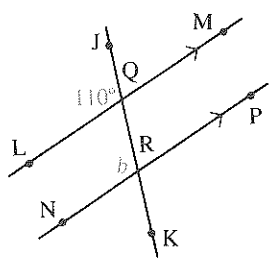


2. Find the angle measure indicated by each letter.

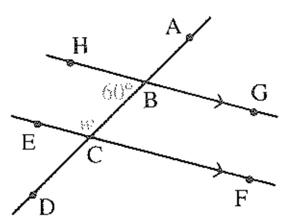
a)



b)

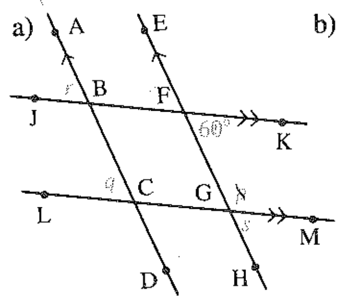


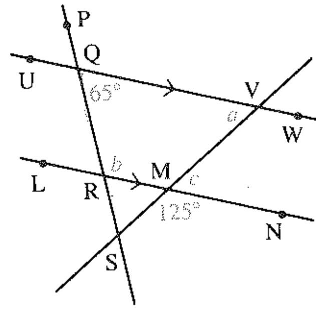
c)

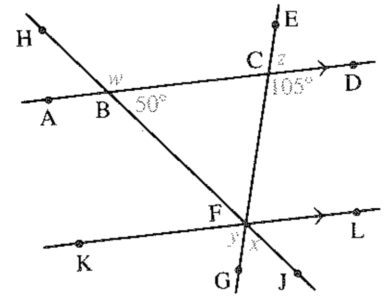


(B)

3. Find the angle measure indicated by each letter.

a) 

b) 

c) 

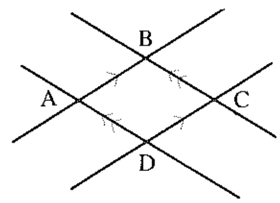
4. Can two intersecting lines both be parallel to a third line? Draw a diagram to support your answer.

5. Can two intersecting lines both be perpendicular to a third line in the same plane? Draw a diagram to support your answer.

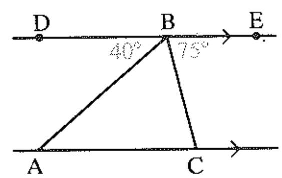
6. In how many points can three lines intersect?

(C)

7. a) Use the diagram to help you explain how you know that in parallelogram ABCD, ∠ABC = ∠ADC.
   b) Use part a) and a property of interior angles to explain why ∠A = ∠C.



8. From the information given in the diagram, find the measures of the angles in △ABC. What is the sum of their measures?



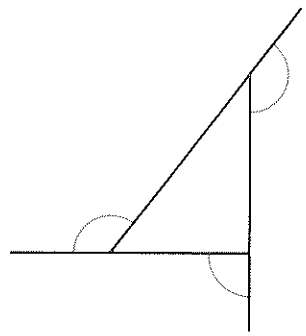
INVESTIGATE

1. Draw a triangle with all of its angles less than 90°. Measure the angles. Add these measurements.
2. Draw a triangle with one angle of 90°. Measure the angles. Add these measurements.
3. Draw a triangle with one angle greater than 90°. Measure the angles. Add these measurements.
4. Do the angles of a triangle appear to have a special property?
5. Write a statement to describe a property of the angles of a triangle.

COMPUTER POWER

The Exterior Angles of a Triangle

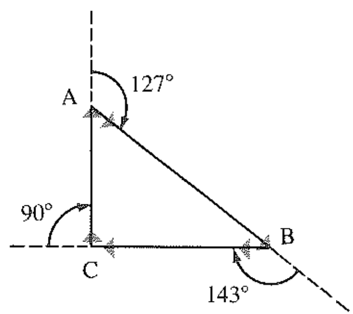
If the sides of a triangle are extended in one direction, an *exterior angle* is formed at each vertex.



In Turtle Geometry, a small "turtle" which looks like this ▲ can be instructed to construct a triangle by a series of *Logo* commands such as these.

Verbal Command	Logo Command
Start at A.	HOME
Turn right 127°.	RT 127
Move forward 100 units (along AB).	FD 100
Turn right 143°.	RT 143
Move forward 80 units (along BC).	FD 80
Turn right 90°.	RT 90
Move forward 60 units (along CA).	FD 60

Graphics Display

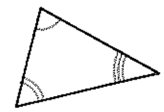
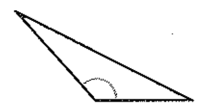


Observe that the turtle turns through the exterior angles as it traces the triangle.

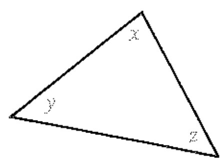
1. What is the total number of degrees through which the turtle turned in drawing the triangle?
2. Write another set of Logo commands that cause the turtle to trace a triangle. (Ensure that the turtle starts and finishes facing upward.) What is the sum of the angles through which your turtle turned?
3. Write a statement about the sum of the three exterior angles of a triangle.
4. What is the sum of the exterior angle and the interior angle at any vertex of a triangle?
5. What is the sum of all three exterior and all three interior angles of a triangle?
6. Use your answers to *Questions 3* and *5* to make a statement about the sum of three interior angles of a triangle.

9-4 ANGLES AND TRIANGLES

Triangles may be classified by the measures of their angles.

Description	**Triangle**	**Example**
all angles are acute	acute triangle	
one angle is 90°	right triangle	
one angle is obtuse	obtuse triangle	

From the previous *INVESTIGATE*, you may have discovered that the sum of the measures of the angles of a triangle is 180°.



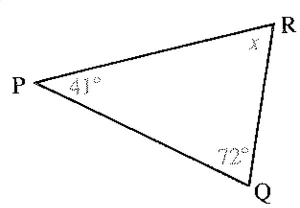
$$x + y + z = 180°$$

For convenience, we delete the phrase "of the measures" in the statement above.

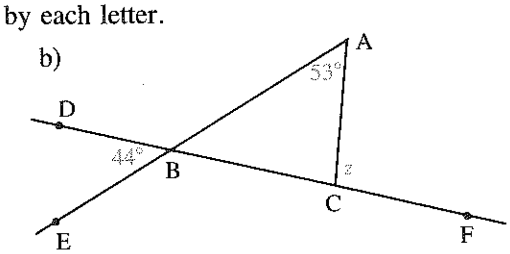
> The sum of the angles in any triangle is 180°.

Example. Find the angle measure indicated by each letter.

a)



b)



Solution. a) Since the sum of the angles in $\triangle PQR$ is 180°,

$$\begin{aligned} x + 41° + 72° &= 180° \\ x + 113° &= 180° \\ x &= 67° \end{aligned}$$

b) Since $\angle ABC$ and $\angle DBE$ are opposite angles,

$$\begin{aligned} \angle ABC &= \angle DBE \\ &= 44° \end{aligned}$$

Since the sum of the angles in $\triangle ABC$ is $180°$,

$$\begin{aligned} \angle ABC + \angle BCA + \angle CAB &= 180° \\ 44° + \angle BCA + 53° &= 180° \\ \angle BCA + 97° &= 180° \\ \angle BCA &= 83° \end{aligned}$$

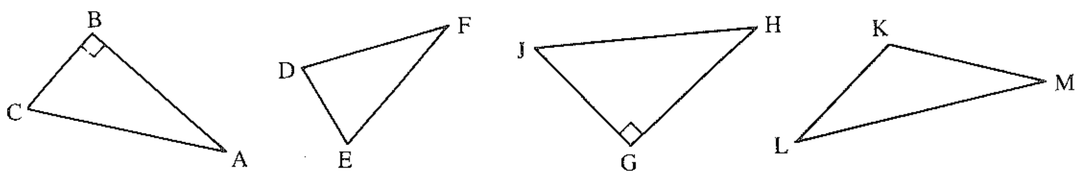
Since $\angle BCF$ is a straight angle,

$$\begin{aligned} \angle BCA + \angle ACF &= 180° \\ 83° + z &= 180° \\ z &= 97° \end{aligned}$$

EXERCISES 9-4

Ⓐ

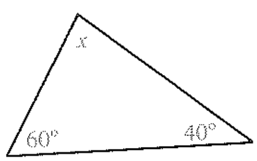
1. Identify each triangle as acute, right, or obtuse.



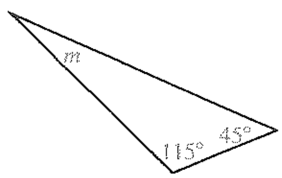
2. Each pair of angles represents the measures of two angles in a triangle. In each case, find the third angle and identify the triangle.
   a) $35°$, $65°$ b) $70°$, $75°$ c) $40°$, $25°$ d) $60°$, $30°$

3. Find the angle measure indicated by each letter.

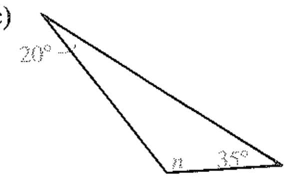
a)



b)



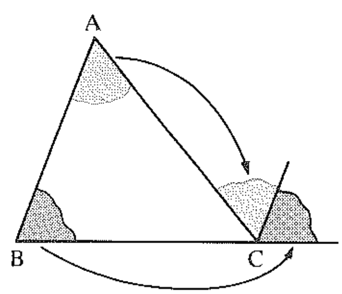
c)



Ⓑ

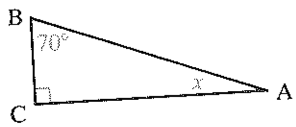
4. Plot each set of points on a grid. Draw and identify each triangle.
   a) $A(4,4)$, $B(7,0)$, $C(0,-2)$
   b) $D(5,0)$, $E(-4,6)$, $F(-3,1)$
   c) $G(-1,6)$, $H(7,4)$, $J(2,1)$
   d) $K(-4,0)$, $L(1,4)$, $M(2,-2)$

5. Cut any triangle from a piece of paper. Label the vertices A, B, and C.
   a) Tear off the corners at A and B and fit them at C, as shown in the diagram. Explain your findings.
   b) Can you fit the angles of a triangle in this way without cutting or tearing?

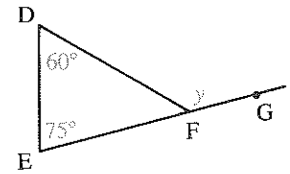


6. Find the angle measure indicated by each letter.

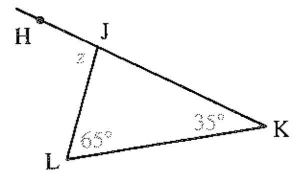
a)



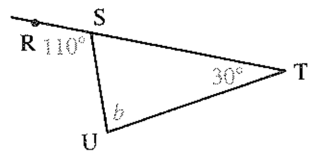
b)



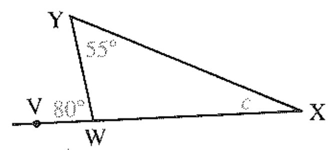
c)



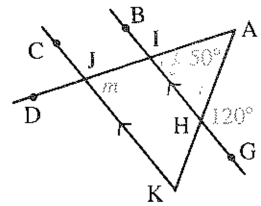
d)



e)



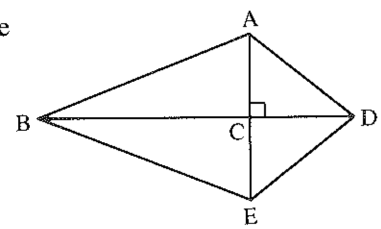
f)



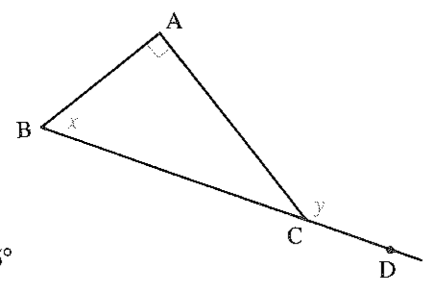
7. State whether a triangle can be drawn having these angles. Give reasons for your answers.
   a) 2 acute angles b) 3 acute angles c) 2 right angles
   d) 2 obtuse angles e) a straight angle f) 1 right angle, 1 obtuse angle

8. Explain why every triangle must have at least two acute angles.

9. a) State how many triangles there are in this figure.
   b) Name those that are:
      i) acute triangles
      ii) right triangles
      iii) obtuse triangles.

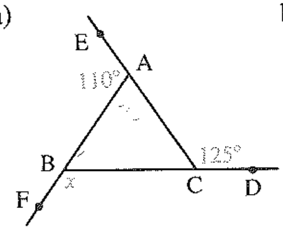


10. In this figure
   a) If the value of x is known, how can the value of y be found?
   b) Find the value of y for each value of x.
      i) $x = 30°$ ii) $x = 50°$ iii) $x = 87°$
   c) If the value of y is known, how can the value of x be found?
   d) Find the value of x for each value of y.
      i) $y = 110°$ ii) $y = 160°$ iii) $y = 175°$
   e) Find an equation relating x and y.

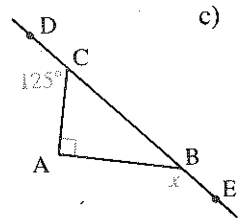


11. Find each value of x.

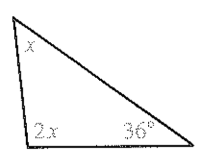
a)



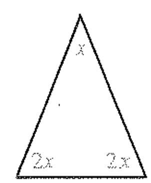
b)



c)



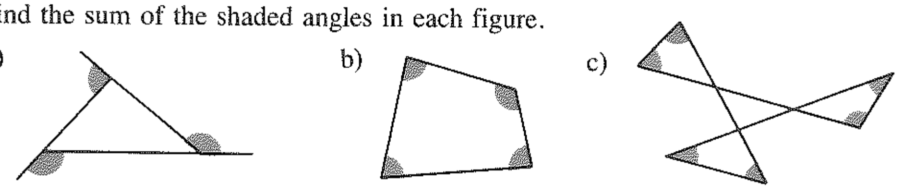
d)



(C)

12. Find the sum of the shaded angles in each figure.

a) b) c)

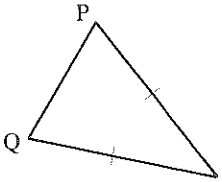


13. What is the sum of the measures of the interior angles of a quadrilateral?

INVESTIGATE

A triangle with at least two sides equal is an isosceles triangle.

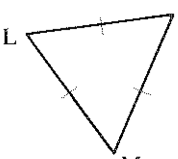
PR = QR



1. Draw an isosceles △PQR using ruler and compasses.
   - Draw two equal line segments PR and QR, with the same end point R.
   - Join PQ.
   - Measure ∠RPQ and ∠PQR.
2. Repeat the procedure with an isosceles triangle of a different size.
3. Do the angles opposite the equal sides of an isosceles triangle appear to have a special property?
4. Write a statement to describe a property of the angles opposite the equal sides of an isosceles triangle.

A triangle with three sides equal is an equilateral triangle.

LM = MN = NL

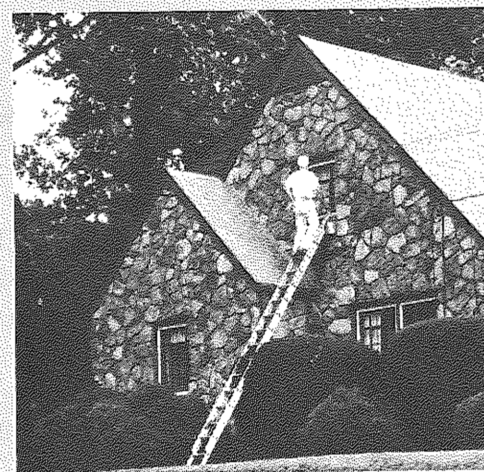


5. Draw an equilateral △LMN using ruler and compasses.
   - Draw any line segment LM.
   - With the distance between compasses point and pencil equal to the length of LM, put compasses point on L and draw an arc. Then put compasses point on M and draw an arc to intersect the first arc.
   - Label the intersection of the arcs, N. Join LN and MN.
   - Measure ∠LMN, ∠MNL, and ∠NLM.
6. Repeat the procedure with an equilateral triangle of a different size.
7. Do the angles of an equilateral triangle appear to have a special property?
8. Write a statement to describe a property of the angles of an equilateral triangle.

PROBLEM SOLVING

Draw a Diagram

A 6.5 m ladder is placed against a wall with the foot of the ladder 2.5 m from the wall. If the top of the ladder slips 0.8 m, how far will the bottom of the ladder slip?

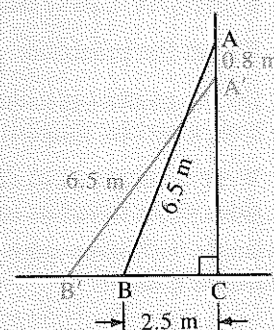


Understand the problem

- How long is the ladder?
- How far from the wall is the top of the ladder? the foot of the ladder?
- How far down the wall does the ladder slip?
- What are you asked to find?

Think of a strategy

- Try drawing a diagram to show the lengths and distances.



Carry out the strategy

- Draw a diagram showing the ladder before slipping (AB) and after slipping (A′B′).
- Mark the known lengths and distances on the diagram.
- We need to find the length BB′. We cannot find this directly. So, find AC, then A′C, then B′C.
- Use the Pythagorean theorem in $\triangle ABC$.

$$\begin{aligned} AB^2 &= BC^2 + AC^2 \\ 6.5^2 &= 2.5^2 + AC^2 \\ AC^2 &= 6.5^2 - 2.5^2 \\ AC &= \sqrt{6.5^2 - 2.5^2} \\ &= 6 \\ A'C &= AC - AA' \\ &= 6 - 0.8 \\ &= 5.2 \end{aligned}$$

Use the Pythagorean theorem in $\triangle A'B'C$.

$$\begin{aligned} A'B'^2 &= B'C^2 + A'C^2 \\ 6.5^2 &= B'C^2 + 5.2^2 \\ B'C^2 &= 6.5^2 - 5.2^2 \\ B'C &= \sqrt{6.5^2 - 5.2^2} \\ &= 3.9 \\ BB' &= B'C - BC \\ &= 3.9 - 2.5 \\ &= 1.4 \end{aligned}$$

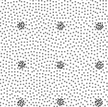
The bottom of the ladder slips 1.4 m.

Look back

- Does a 1.4 m slip seem reasonable for a 0.8 m change in the height of the ladder?

Solve each problem

1. The Cougars lead a league of 5 teams and the Dolphins are last. The Bears are halfway between the Cougars and the Dolphins. If the Eagles are ahead of the Dolphins and the Stallions immediately behind the Bears, name the team that is second in the league.
2. During a game of blindfold bluff, Kevin walks 5 m north, 12 m east, 30 m south-west and 8 m north-west. How far is Kevin now from his starting position?
3. The surface area of a cube is 600 cm^2. The cube is cut into 64 smaller congruent cubes. What is the surface area of one of these cubes?
4. The 40 grade 9 students at Participation H.S. choose to play one or two of the three sports: football, basketball, and hockey. Nineteen play football, 16 play basketball, and 15 play hockey. Included in those numbers are the 6 students who play both football and hockey and 3 who play both football and basketball. How many students play both basketball and hockey?
5. A train leaves at 7:00 A.M. daily from Toronto bound for Vancouver. Simultaneously, another train leaves Vancouver for Toronto. The journey takes exactly 4 days in each direction. If a passenger boards a train in Vancouver, how many Vancouver bound trains will she pass en route to Toronto?
6. How many rectangles can be drawn on a grid of 9 equally spaced dots so that all the vertices are located on dots?



7. A rectangular park is 400 m long and 300 m wide. If it takes 14 min to walk around the perimeter once, how long would it take to walk across a diagonal of the park?
8. How can 10 chairs be arranged along the perimeter of a rectangular room so that there is an equal number of chairs along each wall?
9. The distances between a pine tree, an oak tree, and a maple tree were measured and recorded.

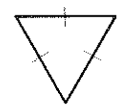
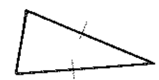
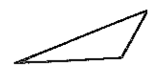
From	**To**	**Distance**
Pine	Maple	150 m
Pine	Oak	100 m
Maple	Oak	45 m

Were the measurements correct? Explain your answer.

10. A rectangular field is twice as long as its width. Its perimeter is less than 1.2 km. What does this tell you about the area of the field?

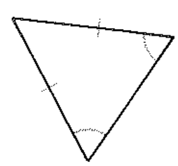
9-5 ISOSCELES AND EQUILATERAL TRIANGLES

In the previous section, triangles were classified according to the measures of their angles. Triangles are also classified according to the lengths of their sides.

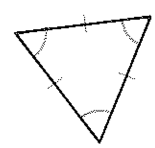
Description	Triangle	Example
3 sides equal	equilateral triangle	
at least 2 sides equal	isosceles triangle	
no sides equal	scalene triangle	

From the previous *INVESTIGATE*, you may have discovered these properties of isosceles triangles and equilateral triangles.

In an isosceles triangle, the angles opposite the equal sides are equal.

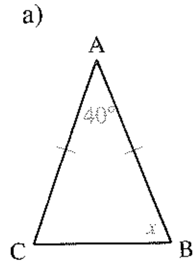


In an equilateral triangle, the angles are equal and have a measure of 60°.

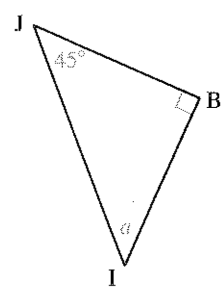


Example 1. Find the angle measure indicated by each letter.

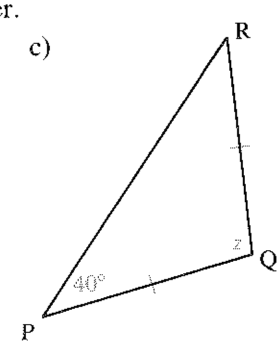
a)



b)



c)



Solution. a) Since AB = AC, then $\angle ACB = x$
Since the sum of the angles in $\triangle ABC$ is 180°,

$$\begin{aligned} \angle ABC + \angle BCA + \angle CAB &= 180^\circ \\ x + x + 40^\circ &= 180^\circ \\ 2x + 40^\circ &= 180^\circ \\ 2x &= 140^\circ \\ x &= 70^\circ \end{aligned}$$

b) Since the sum of the angles in $\triangle BJI$ is 180°,

$$\begin{aligned} \angle BJI + \angle JIB + \angle IBJ &= 180^\circ \\ 45^\circ + a + 90^\circ &= 180^\circ \\ a + 135^\circ &= 180^\circ \\ a &= 45^\circ \end{aligned}$$

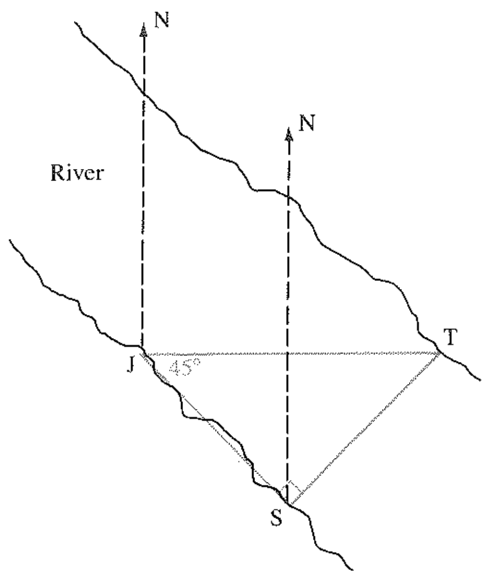
c) Since PQ = QR, $\angle PRQ = 40^\circ$
Since the sum of the angles in $\triangle PQR$ is 180°,

$$\begin{aligned} \angle PQR + \angle QRP + \angle RPQ &= 180^\circ \\ z + 40^\circ + 40^\circ &= 180^\circ \\ z + 80^\circ &= 180^\circ \\ z &= 100^\circ \end{aligned}$$

Example 2. Sami and Jane have only a measuring tape and a compass. How can they measure the width of a river that they cannot cross?

Solution. Sami and Jane draw a diagram to help them solve the problem.

On their side of the river, they mark a point S, opposite a large tree T on the far bank. ST represents the width of the river. After taking the compass bearing of T from S, Jane walks in a line perpendicular to ST. She walks until she reaches a point J, where the compass reading indicates that $\angle SJT = 45^\circ$.



Since the sum of the angles in $\triangle SJT$ is 180°,

$$\begin{aligned} \angle SJT + \angle JTS + \angle TSJ &= 180^\circ \\ 45^\circ + \angle JTS + 90^\circ &= 180^\circ \\ \angle JTS &= 45^\circ \end{aligned}$$

Since $\angle JTS = \angle SJT = 45^\circ$, $\triangle SJT$ is isosceles.
Hence, SJ = ST

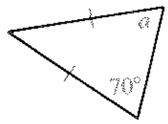
Sami measures the distance SJ. This is the approximate width of the river.

EXERCISES 9-5

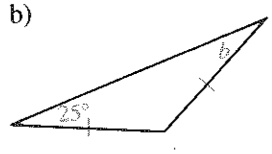
Ⓐ

1. Find the angle measure indicated by each letter.

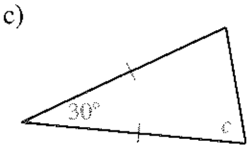
a)



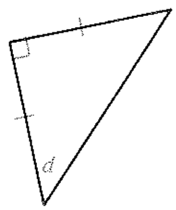
b)



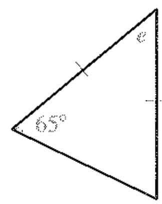
c)



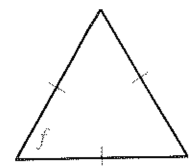
d)



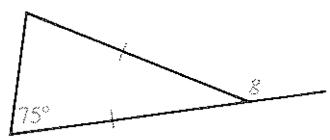
e)



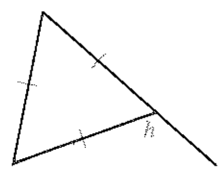
f)



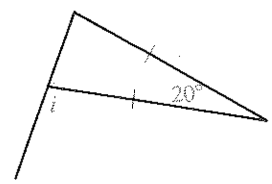
g)



h)



i)

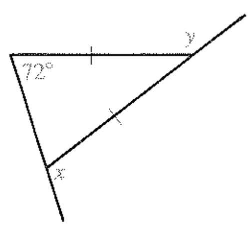


2. a) Are all equilateral triangles isosceles? b) Are all isosceles triangles equilateral?

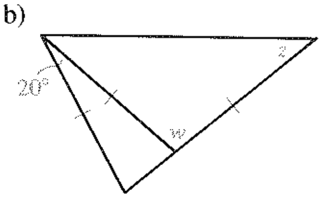
Ⓑ

3. Find the angle measure indicated by each letter.

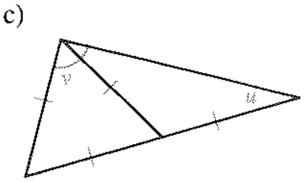
a)



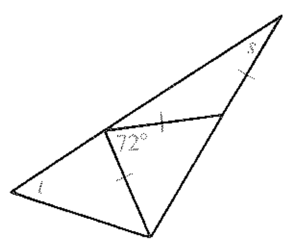
b)



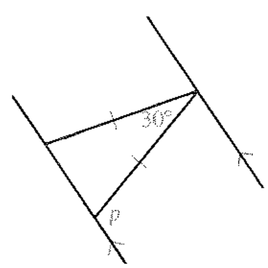
c)



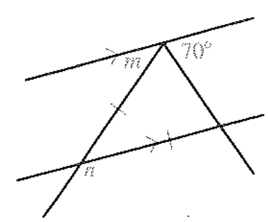
d)



e)

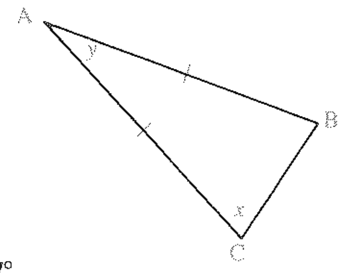


f)

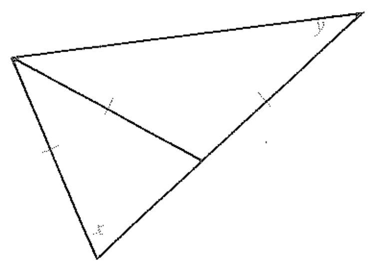


4. Draw an example of each triangle.
   a) an isosceles right triangle
   b) an isosceles obtuse triangle
   c) a scalene right triangle
   d) a scalene obtuse triangle
   e) an isosceles acute triangle
   f) a scalene acute triangle

5. In isosceles $\triangle ABC$
   a) If the value of x is known, how can the value of y be found?
   b) Find the value of y for each value of x.
      i) $x = 70°$ ii) $x = 25°$ iii) $x = 43°$
   c) If the value of y is known, how can the value of x be found?
   d) Find the value of x for each value of y.
      i) $y = 80°$ ii) $y = 110°$ iii) $y = 17°$
   e) Find an equation relating x and y.

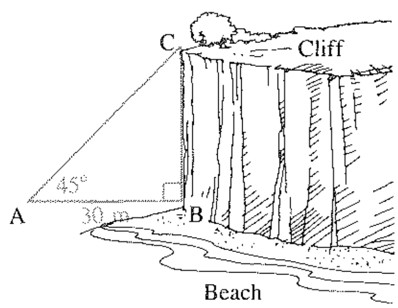


6. In this figure
   a) If the value of x is known, how can the value of y be found?
   b) Find the value of y for each value of x.
      i) $x = 60°$ ii) $x = 40°$ iii) $x = 26°$
   c) If the value of y is known, how can the value of x be found?
   d) Find the value of x for each value of y.
      i) $y = 40°$ ii) $y = 25°$ iii) $y = 81°$
   e) Find an equation relating x and y.

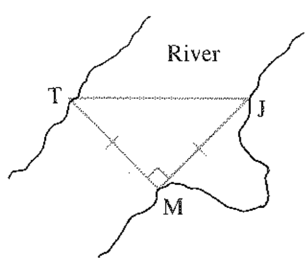


7. Amanda wanted to find the height of a cliff. She drew a diagram in which BC represented this height.

   Amanda had a clinometer for measuring $\angle CAB$. She moved away from the base of the cliff to a point A, where the clinometer showed $\angle CAB$ as 45°. Amanda measured the distance AB as 30 m. How high was the cliff?



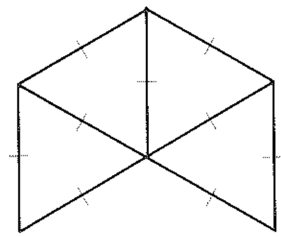
8. Michael and Julie are located at M and J respectively. A large tree is located at T on the opposite bank of the river. The distance MJ cannot be measured with the tape. Determine how Michael and Julie could find the approximate width MT of the river.



9. The measure of one angle of an isosceles triangle is given. Find the possible measures of the other angles.
   a) 30° b) 40° c) 80° d) 90° e) 110°

Ⓒ

10. A number of equilateral triangles are joined together with whole sides touching. The diagram shows a figure formed with four equilateral triangles. Find how many different figures can be formed for each number of equilateral triangles used.
    a) 3 b) 4 c) 5



11. Explain how to find the height of a tree with a right, isosceles, plastic triangle and a measuring tape.

INVESTIGATE

1. Draw $\triangle ABC$ with $AB = 6$ cm, $BC = 8$ cm, and $CA = 12$ cm.
   - Draw line segment AB 6 cm long.
   - With compasses point on B, and radius 8 cm, drawn an arc.
   - With compasses point on A, and radius 12 cm, draw an arc to intersect the first arc. Label the point of intersection C.
   - Draw BC and CA.

   Can you draw $\triangle PQR$ with $PQ = 6$ cm, $QR = 8$ cm, and $RP = 12$ cm such that its size and shape are different from $\triangle ABC$?

2. Draw $\triangle ABC$ with $\angle A = 65°$, $\angle B = 85°$, and $\angle C = 30°$. Can you draw $\triangle PQR$ with $\angle P = 65°$, $\angle Q = 85°$, and $\angle R = 30°$ such that its size and shape are different from $\triangle ABC$?

3. Draw $\triangle ABC$ with $AB = 8$ cm, $BC = 5$ cm, and $\angle B = 50°$. Can you draw $\triangle PQR$ with $PQ = 8$ cm, $QR = 5$ cm, and $\angle Q = 50°$ such that its size and shape are different from $\triangle ABC$?

4. Draw $\triangle ABC$ with $AB = 7$ cm, $AC = 4$ cm, and $\angle B = 30°$. Can you draw $\triangle PQR$ with $PQ = 7$ cm, $PR = 4$ cm, and $\angle Q = 30°$ such that its size and shape are different from $\triangle ABC$?

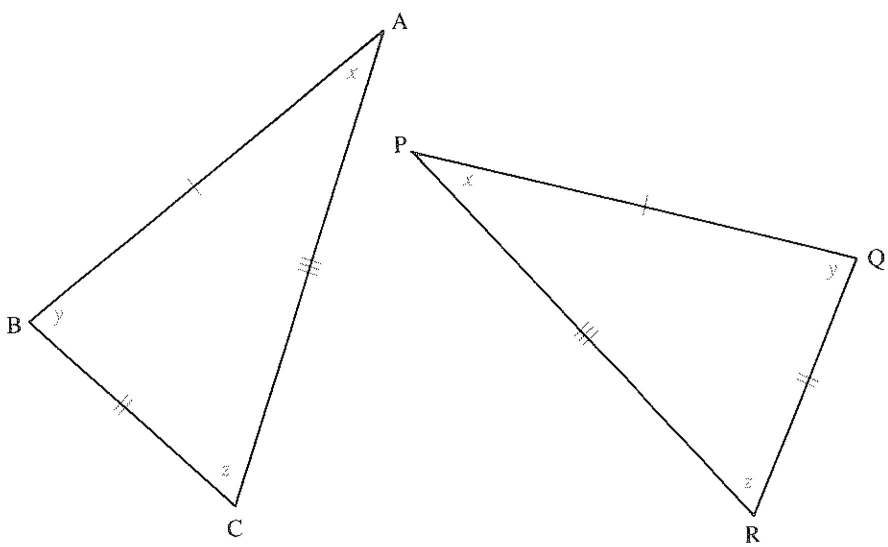
5. Draw $\triangle ABC$ with $\angle A = 62°$, $\angle B = 80°$, and $BC = 6$ cm. Can you draw $\triangle PQR$ with $\angle P = 62°$, $\angle Q = 80°$, and $QR = 6$ cm such that its size and shape are different from $\triangle ABC$?

Which measurements of a triangle would you have to be given so that only one triangle could be drawn?

9-6 CONGRUENT TRIANGLES

If two geometric figures have the same size and shape, they are said to be *congruent*.

Two congruent triangles can be made to coincide because their corresponding sides and corresponding angles are equal.



In $\triangle ABC$ and $\triangle PQR$

$\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$

$AB = PQ$, $BC = QR$, $CA = RP$

Therefore, $\triangle ABC \cong \triangle PQR$

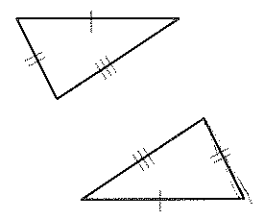
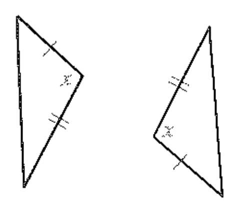
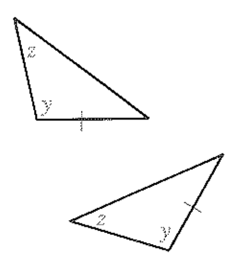
The sign $\cong$ is read "is congruent to".

The order in which the vertices are listed when pairs of triangles are congruent indicates which angles and sides correspond.
$\triangle ABC \cong \triangle PQR$

If the three sides and the three angles of two triangles are equal, then the triangles are congruent. However, it is not necessary to know this much information to show that two triangles are congruent.

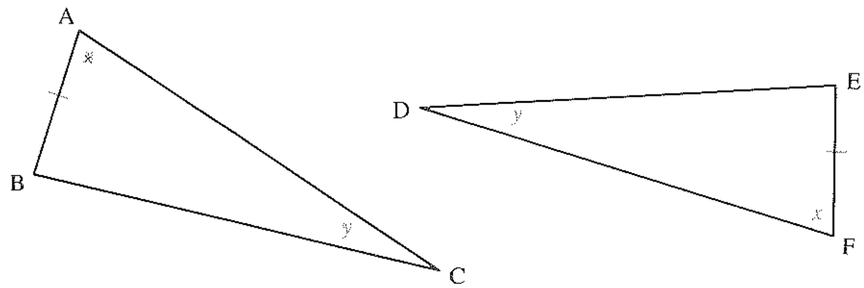
From the previous *INVESTIGATE*, you may have discovered that given certain measurements of a triangle, there is only one possible triangle that can be drawn. This means that if two triangles are each described by this same set of measurements, those triangles must be equal in size and shape, and hence congruent.

The table shows three conditions. Any one of these conditions is sufficient to show that two triangles are congruent.

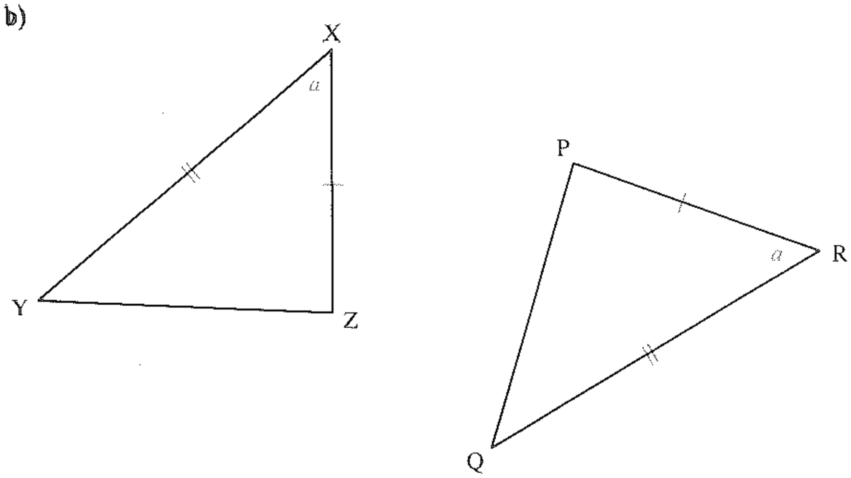
Conditions for Congruence	**Illustration**	**Abbreviation**
Three sides of one triangle are respectively equal to three sides of another triangle.		SSS (side, side, side)
Two sides and the contained angle of one triangle are respectively equal to two sides and the contained angle of another triangle.		SAS (side, angle, side)
Two angles and one side of one triangle are respectively equal to two corresponding angles and the corresponding side of another triangle.		AAS (angle, angle, side)

Example 1. For each pair of triangles, explain why they are congruent and state the condition for congruence. List the equal sides and the equal angles.

a)



b)



Solution. a) In $\triangle ABC$ and $\triangle DEF$

$\angle A$ and $\angle F$ are equal.
Side AB and side FE are equal.
$\angle C$ and $\angle D$ are equal.
Therefore, $\triangle ABC \cong \triangle FED$ AAS

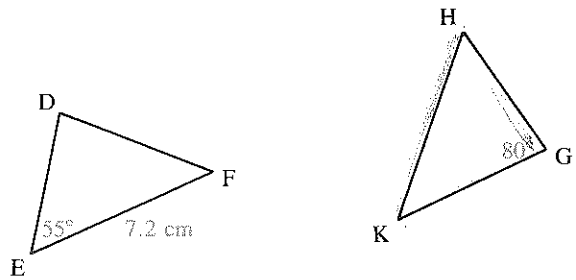
Since $\triangle ABC$ is congruent to $\triangle FED$,
$BC = ED$, $AC = FD$, and $\angle B = \angle E$

b) In $\triangle XYZ$ and $\triangle PQR$

$\angle X$ and $\angle R$ are equal.
Side XZ and side RP are equal.
Side XY and side RQ are equal.
Therefore, $\triangle XYZ \cong \triangle RQP$ SAS

Since $\triangle XYZ$ is congruent to $\triangle RQP$,
$YZ = QP$, $\angle Y = \angle Q$, and $\angle Z = \angle P$

Example 2. $\triangle DEF$ is congruent to $\triangle GHK$.
a) Find the length of HK.
b) Find the measures of the unmarked angles in the triangles.

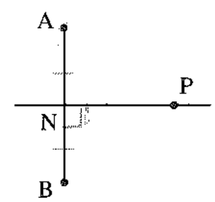


Solution. a) Since $\triangle DEF$ is congruent to $\triangle GHK$, EF = HK.
But EF = 7.2 cm, so HK = 7.2 cm

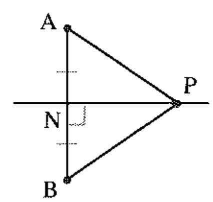
b) Since $\triangle DEF \cong \triangle GHK$
$\angle E = \angle H$, so $\angle H = 55°$
$\angle G = \angle D$, so $\angle D = 80°$
$\angle F = \angle K$
But $\angle F = 180° - 80° - 55°$
$= 45°$
So $\angle K = 45°$

The conclusions from showing that two triangles are congruent may be used to develop other geometric properties.

Example 3. P is any point on the perpendicular bisector of line segment AB.
a) Explain why $\triangle PNA \cong \triangle PNB$.
b) Explain why PA = PB.
c) State a conclusion about any point on the perpendicular bisector of a line segment.



Solution. a) Join PA and PB.
In $\triangle PAN$ and $\triangle PBN$
AN and BN are equal.
$\angle PNA$ and $\angle PNB$ are both equal to 90°.
PN is a common side to both triangles.
Therefore, $\triangle PNA \cong \triangle PNB$ SAS



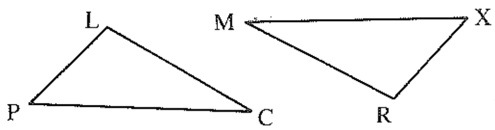
b) Since $\triangle PNA \cong \triangle PNB$, corresponding sides are equal.
PA and PB are corresponding sides.
Hence, PA = PB

c) Any point on the perpendicular bisector of a line segment is equidistant from the ends of the line segment.

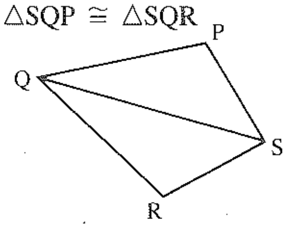
EXERCISES 9-6

Ⓐ

1. Name the equal angles and the equal sides in each figure.
   a) $\triangle MRX \cong \triangle CLP$



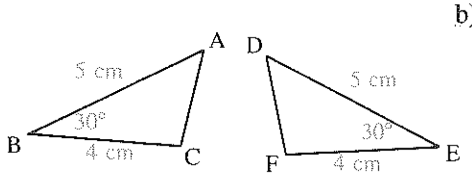
   b) $\triangle SQP \cong \triangle SQR$



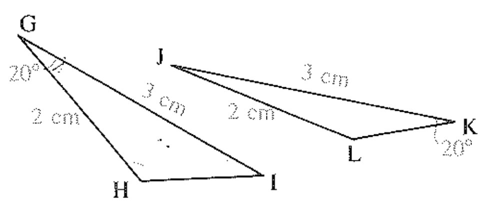
2. Name the equal sides and the equal angles.
   Figure PQRS $\cong$ figure XWZY

3. State which pairs of triangles are congruent. For those triangles that are congruent, state the condition for congruence.

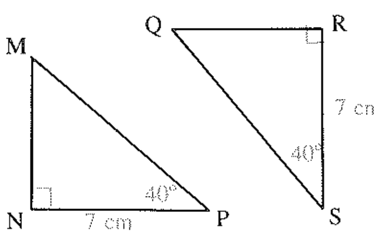
a)



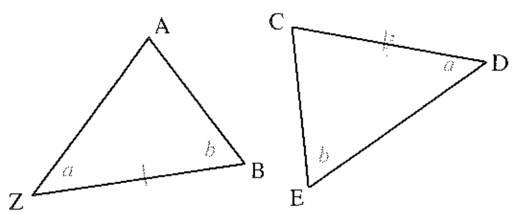
b)



c)



d)



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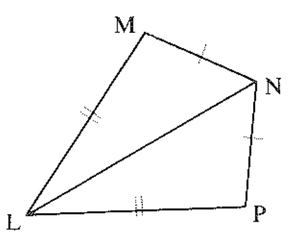

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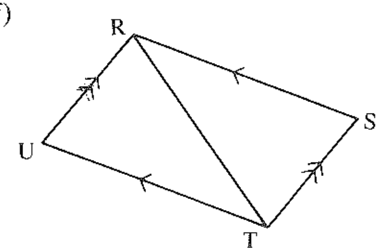
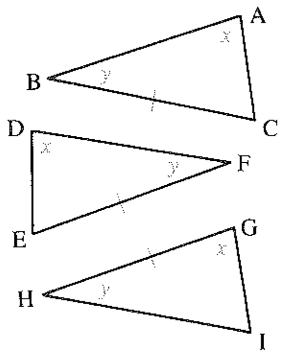


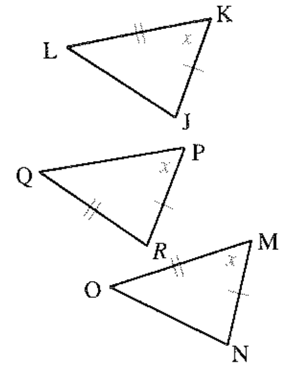
B

4. Find pairs of congruent triangles and state the condition for congruence.

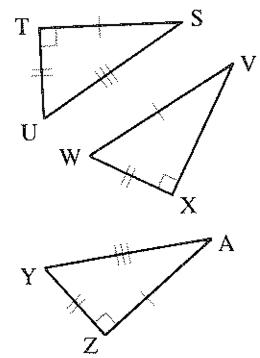
a)



b)

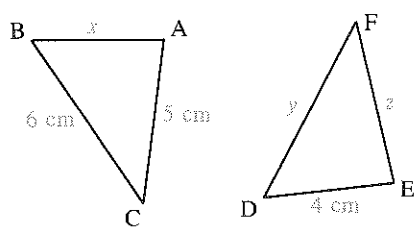


c)

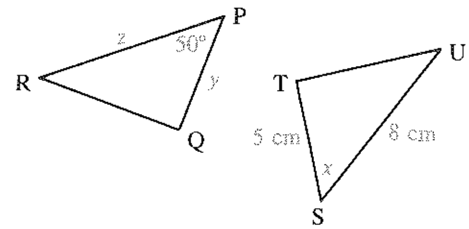


5. Find the value indicated by each letter.

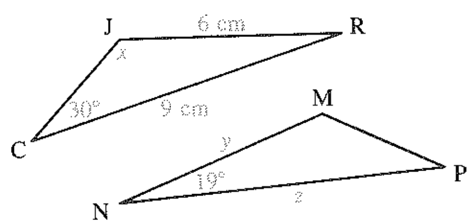
a) $\triangle ABC \cong \triangle DEF$



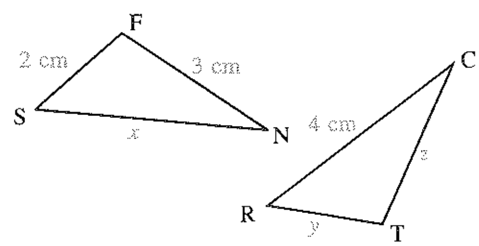
b) $\triangle PQR \cong \triangle STU$



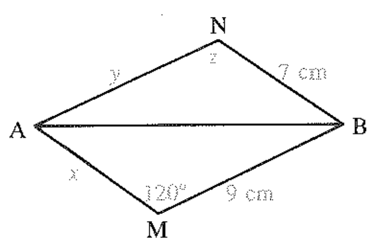
c) $\triangle JRC \cong \triangle MNP$



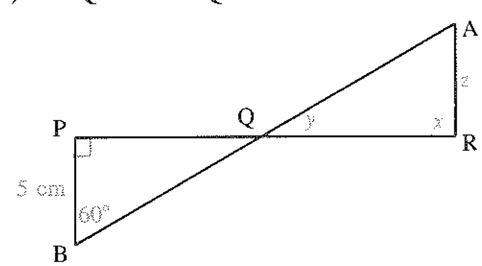
d) $\triangle FSN \cong \triangle TRC$



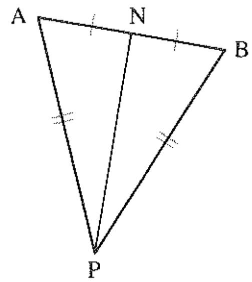
e) $\triangle BNA \cong \triangle AMB$



f) $\triangle PQB \cong \triangle RQA$



6. N is the midpoint of line segment AB. P is any point such that PA = PB.
   a) Explain why $\triangle PNA \cong \triangle PNB$.
   b) Explain why $\angle PNA = \angle PNB = 90°$.
   c) State a conclusion about any point that is equidistant from the endpoints of a line segment.



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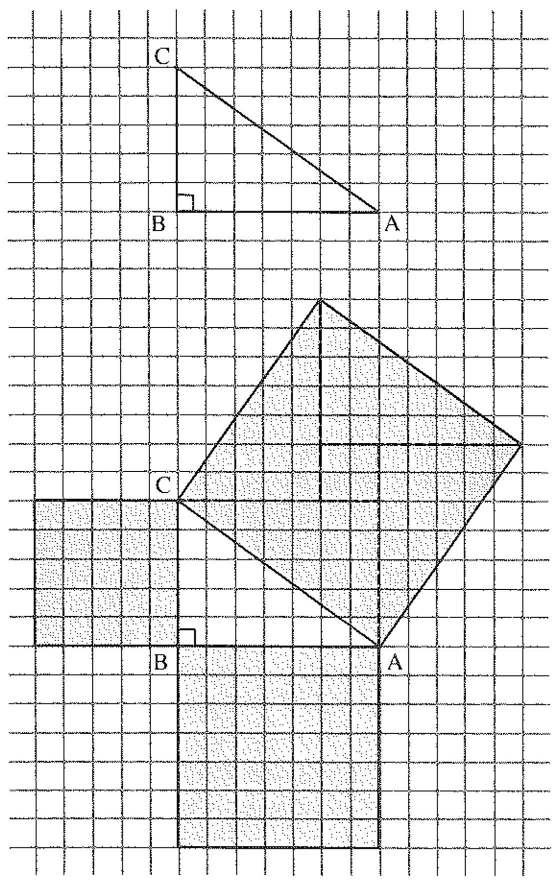
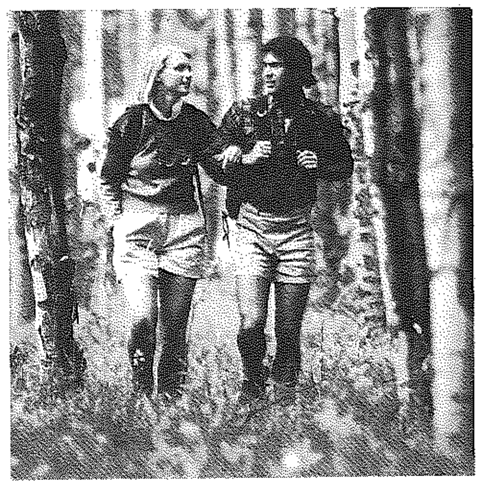
7. Explain why any point on the bisector of an angle is equidistant from the sides of the angle.

INVESTIGATE

1. Draw a right triangle on squared paper, so that the arms of the right angle lie along the lines of the paper.
2. Draw a square on each side of the triangle.
3. Find the area of each square. To find the area of the square on the hypotenuse (the longest side), you can count the small squares and parts of squares. Alternatively, you can divide the square into right triangles and use the formula for the area of a triangle.
4. Repeat *Questions 1* to *3* for different right triangles. What do you notice about the areas of the squares?
5. Write a statement to describe a property of the areas of the squares drawn on the sides of a right triangle.

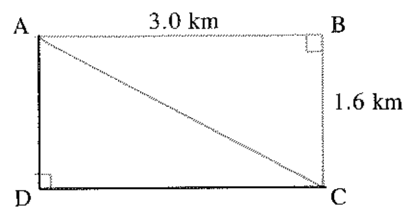




9-7 THE PYTHAGOREAN THEOREM

Anna and Lim are on a hike. They come to a field that is rectangular and measures 3.0 km by 1.6 km. Anna decides to take a short cut and walk diagonally across the field. Lim walks around two sides of the field. Who walks farther and by how much?

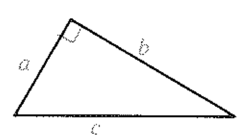
To find the distances walked, draw a diagram of the field.



Lim walks (3.0 km + 1.6 km) or 4.6 km.
Anna walks along the path represented by AC. To find the length of the path, use the Pythagorean theorem. Recall that this theorem states that, for a right triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides. You may have discovered this result in the previous *INVESTIGATE*.

Since the area of a square is equal to the square of the length of a side, the Pythagorean theorem is usually stated in terms of the lengths of the sides of a right triangle.

Pythagorean Theorem
For any right triangle with sides of lengths a, b, and c, where c is the hypotenuse, $c^2 = a^2 + b^2$



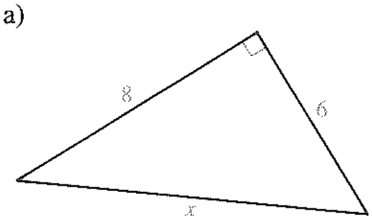
We can apply the Pythagorean Theorem in $\triangle ABC$, on the facing page.

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= 3.0^2 + 1.6^2 \\ &= 11.56 \\ AC &= \sqrt{11.56} \\ &= 3.4 \end{aligned}$$

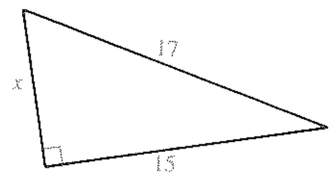
Anna walks 3.4 km.
The difference between the distances walked is (4.6 km − 3.4 km) or 1.2 km.
Lim walks 1.2 km farther than Anna.

Example 1. Calculate each value of x.

a)



b)



Solution. a)

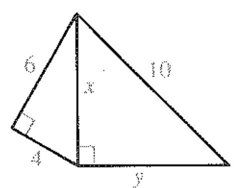
$$\begin{aligned} x^2 &= 6^2 + 8^2 \\ &= 36 + 64 \\ &= 100 \\ x &= \sqrt{100} \\ &= 10 \end{aligned}$$

b)

$$\begin{aligned} 17^2 &= x^2 + 15^2 \\ 289 &= x^2 + 225 \\ 289 - 225 &= x^2 \\ 64 &= x^2 \\ x &= \sqrt{64} \\ &= 8 \end{aligned}$$

When taking the square root, ignore the negative root because length cannot be negative.

Example 2. Find the length, to one decimal place, indicated by each letter.



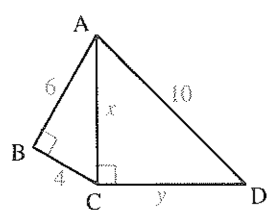
Solution. Label the diagram.
Use the Pythagorean theorem in each triangle.

In $\triangle ABC$

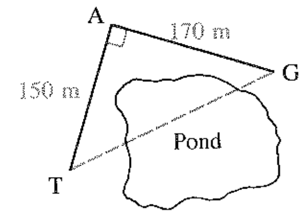
$$\begin{aligned} x^2 &= 6^2 + 4^2 \\ &= 52 \\ x &= \sqrt{52} \\ &\doteq 7.2 \end{aligned}$$

In $\triangle ACD$

$$\begin{aligned} 10^2 &= x^2 + y^2 \\ 100 &= 52 + y^2 \\ 48 &= y^2 \\ y &= \sqrt{48} \\ &\doteq 6.9 \end{aligned}$$



Example 3. The 12th hole at Sandy Dunes golf club is a right-angled "dog leg". What is the distance TG from the tee to the green?



Solution. It follows from the Pythagorean theorem that

$$\begin{aligned} TG^2 &= TA^2 + AG^2 \\ &= 150^2 + 170^2 \\ &= 51\ 400 \\ TG &= \sqrt{51\ 400} \\ &\doteq 226.7 \end{aligned}$$

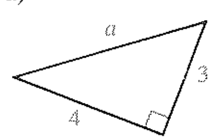
The distance from the tee to the green is approximately 227 m.

EXERCISES 9-7

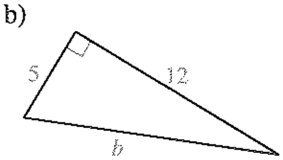
Ⓐ

1. Find the length indicated by each letter.

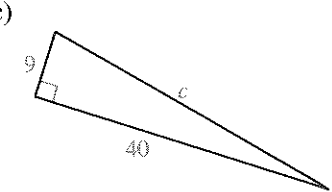
a)



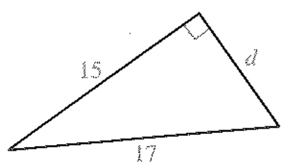
b)



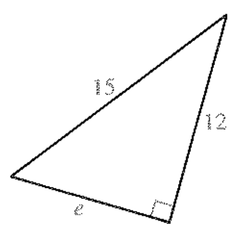
c)



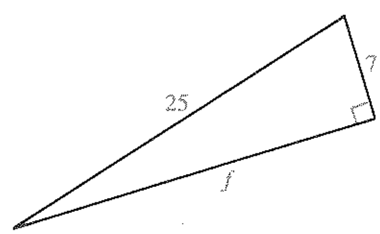
d)



e)

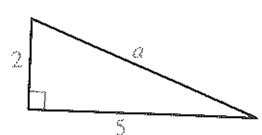


f)

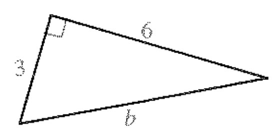


2. Find the length, to one decimal place, indicated by each letter.

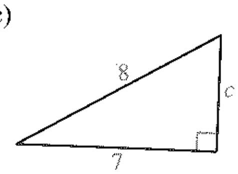
a)



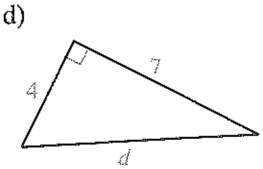
b)



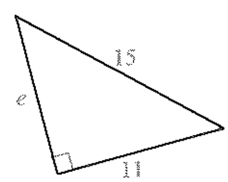
c)



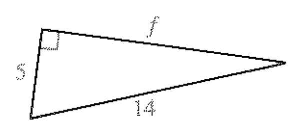
d)



e)



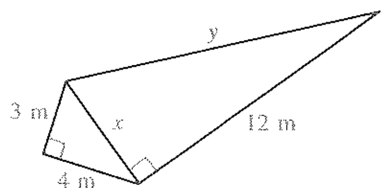
f)



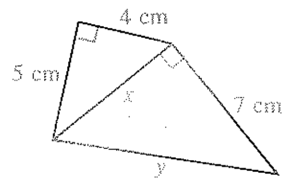
(B)

3. Find the length, to one decimal place, indicated by each letter.

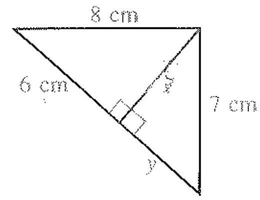
a)



b)



c)



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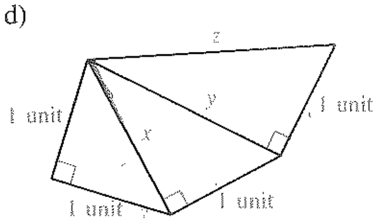

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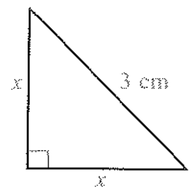
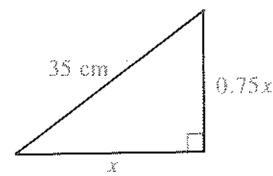
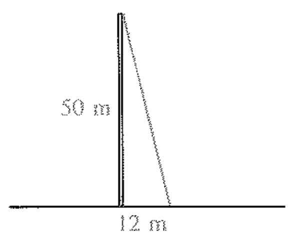


f)

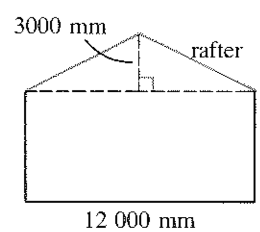


4. A guy wire is attached 50 m up a tower and 12 m from its base. Find the length of the guy wire to the nearest tenth of a metre.



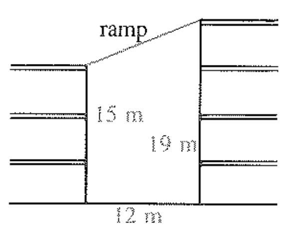
5. A ladder, 8.2 m long, is placed with its foot 1.8 m from a wall. How high up the wall will the ladder reach? Give the answer to the nearest tenth of a metre.

6. Find the length of the rafters for a building, which is 12 000 mm wide and has the peak of the roof 3000 mm above the ceiling. Give the answer to the nearest centimetre.



7. Can an umbrella 1.3 m long be packed flat in a box 1.1 m by 0.3 m? Give reasons for your answer.

8. A ramp is to be built from the top level of one parking garage to another. Calculate the length of the ramp to the nearest tenth of a metre.



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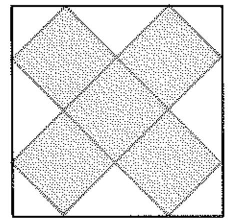
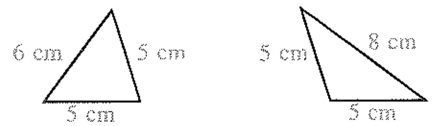
9. Each colored square has sides 3 cm long.
   a) Find the lengths of the sides of the outer square, to the nearest millimetre.
   b) What percent of the outer square is covered by the five colored squares?



10. A TV set has a screen with a diagonal of length 66 cm. If the screen is 1.2 times as wide as it is high, find its width and height.

11. a) Show that the two isosceles triangles have the same area.
    b) Find another pair of isosceles triangles that have equal areas.



CALCULATOR POWER

The Pythagorean Theorem

The evaluation of expressions such as $6^2 + 4^2$ (*Example 2*, page 339) is simplified with the use of a calculator.
For a scientific calculator, here is a possible keying sequence.
Key in: $\boxed{6}\ \boxed{x^2}\ \boxed{+}\ \boxed{4}\ \boxed{x^2}\ \boxed{=}\ \boxed{\sqrt{\ }}$ to display 7.2111026
If the calculator does not have a $\boxed{\sqrt{\ }}$ key, this symbol usually occurs above the $\boxed{x^2}$ key. Here is the keying sequence in this situation.
Key in: $\boxed{6}\ \boxed{x^2}\ \boxed{+}\ \boxed{4}\ \boxed{x^2}\ \boxed{=}\ \boxed{\text{INV}}\ \boxed{x^2}$

A 4-function calculator can be used to evaluate the expression. However, if the numbers are keyed in as they appear, $\boxed{6}\ \boxed{\times}\ \boxed{6}\ \boxed{+}\ \boxed{4}\ \boxed{\times}\ \boxed{4}\ \boxed{=}\ \boxed{\sqrt{\ }}$, the result may be 12.649 111, which is wrong. If this happened, the calculator added the first 4 to the result of 6 times 6 before multiplying by the second 4.

This problem can be solved by rewriting the expression before keying it in.

$$6^2 + 4^2 = \left(\frac{6^2}{4} + 4\right)4$$

Key in: $\boxed{6}\ \boxed{\times}\ \boxed{6}\ \boxed{\div}\ \boxed{4}\ \boxed{+}\ \boxed{4}\ \boxed{\times}\ \boxed{4}\ \boxed{=}\ \boxed{\sqrt{\ }}$, which gives the correct result.

The problem can also be solved by using the memory.

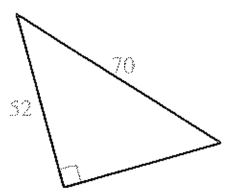
COMPUTER POWER

The Pythagorean Theorem

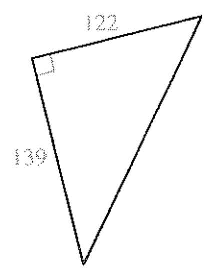
You can use the following program to find the length of any side of a right triangle, given the lengths of the other two sides.

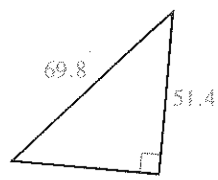
```
100 REM *** PYTHAGOREAN THEOREM ***
110 INPUT "DO YOU KNOW THE HYPOTENUSE (Y OR N)? ";Z$
120 IF Z$ = "Y" THEN 170
130 PRINT "ENTER THE LENGTHS OF THE SIDES"
140 INPUT "SEPARATED BY A COMMA: ";X,Y
150 Z = SQR (X * X + Y * Y)
160 PRINT "THE HYPOTENUSE IS: ";Z: GOTO 220
170 INPUT "ENTER THE HYPOTENUSE LENGTH: ";Z
180 INPUT "ENTER THE LENGTH OF THE OTHER SIDE: ";X
190 Y = SQR (Z * Z - X * X)
200 PRINT "THE LENGTH OF THE THIRD SIDE IS:"
210 PRINT Y
220 INPUT "PRESS S TO STOP, RETURN TO CONTINUE: ";Y$
230 IF Y$ <> "S" THEN 110
240 END
```

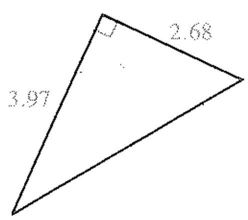
1. Use the program to find the length of the third side of each triangle.

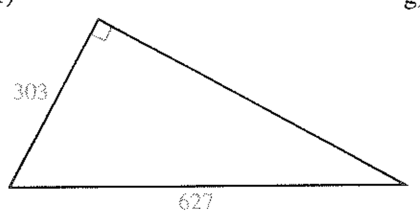
a)


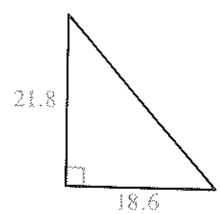
b)


c)


d)


e)


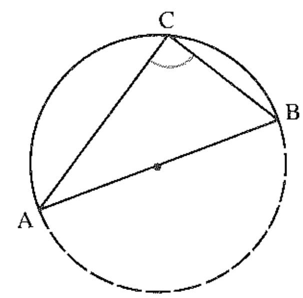
f)


g)


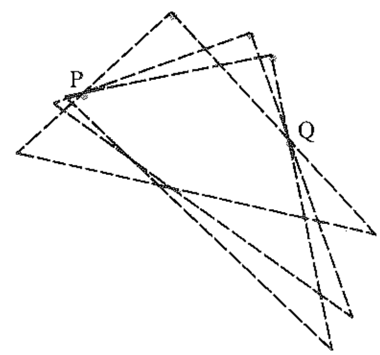
2. Find the third side of $\triangle ABC$, where $\angle ABC = 90°$.
   a) AB = 2.37 cm, BC = 4.19 cm b) AB = 7.66 m, BC = 7.66 m
   c) BC = 44.9 cm, AC = 59.3 cm d) AB = 1.92 m, AC = 3.06 m

INVESTIGATE

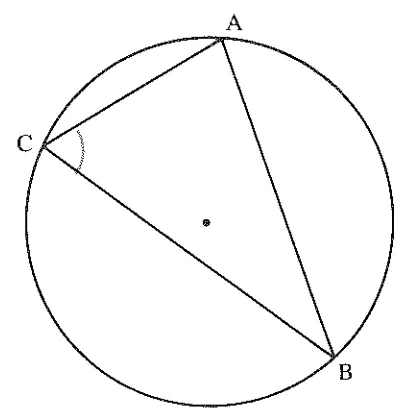
Angles in a Circle



1. a) Draw a semicircle with diameter AB.
   b) Mark a point C on the semicircle. Join AC and BC.
   c) Measure ∠C.
   d) Repeat parts b) and c) for other positions of C on the semicircle. What do you notice?
   e) Repeat parts a) to d) for other semicircles.
   f) Write a statement to describe a property of angles in a semicircle.

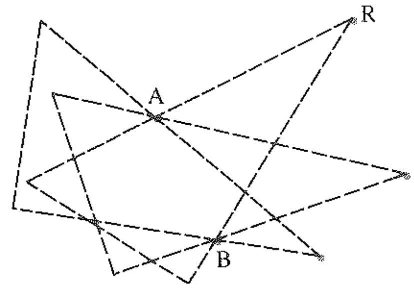


2. a) Mark two points P and Q.
   b) Use a plastic or cardboard right triangle. Position the triangle so that the points P and Q lie on the shorter sides of the triangle. Mark the position of the right angle.
   c) Repeat part b) several times and mark the different positions of the right angles.
   d) What do you notice about the dots marking the positions of the right angle?

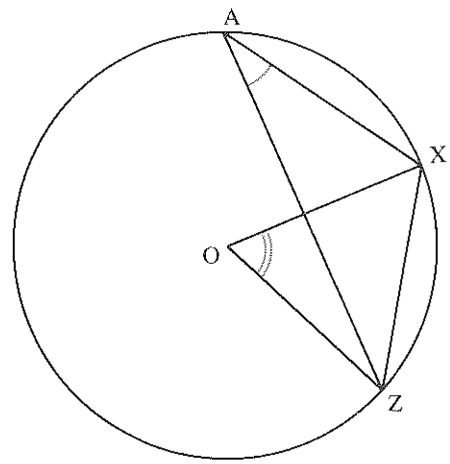


3. a) Draw a circle with any chord AB.
   b) Mark a point C on the circle. Join AC and BC.
   c) Measure ∠C.
   d) Repeat parts b) and c) for other positions of C on the circle. What do you notice?
   e) Repeat parts a) to d) for other circles.
   - ∠C is called an inscribed angle. An angle is *inscribed* in a circle when its vertex is on the circle and its sides are chords.
   f) Write a statement to describe a property of angles inscribed in a circle.

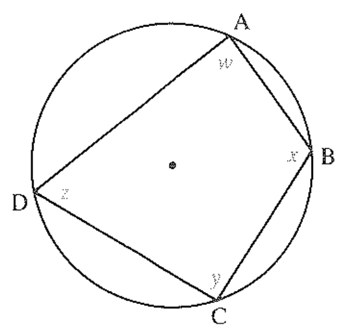
4. a) Mark two points A and B.
   b) Use a plastic or cardboard triangle.
   c) Choose one angle of the triangle and label it R. Position the triangle so that the points A and B lie on the arms of ∠R. Mark the position of ∠R.
   d) Repeat part c) several times and mark the different positions of ∠R.
   e) What do you notice about the dots marking the positions of ∠R?



5. a) Draw a circle with any chord XZ.
   b) Mark the centre of the circle O and any point A on the circumference.
   c) Join OX, OZ, AX, and AZ.
   d) Measure ∠XAZ and ∠XOZ. What do you notice?
   e) Repeat parts a) to d) for other circles.
   - We say that ∠XAZ and ∠XOZ are subtended by the chord XZ. A chord *subtends* an angle when the angle is formed by line segments drawn from the ends of the chord.
   f) Write a statement to describe a property of angles in a circle, when one angle is at the centre of the circle, one angle is at the circumference, and both angles are subtended by the same chord.

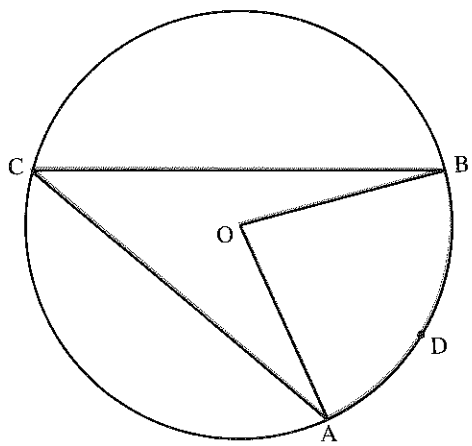


6. a) Draw a circle and mark 4 points A, B, C, and D on the circumference.
   b) Join AB, BC, CD, and DA.
   c) Measure ∠ABC and ∠ADC. What do you notice?
   d) Measure ∠BAD and ∠BCD. What do you notice?
   e) Repeat parts a) to d) for other circles.
   f) Write a statement to describe a property of opposite angles of a quadrilateral, when the quadrilateral has its vertices on the circumference of a circle.



9-8 ANGLES IN A CIRCLE

To learn about angles in a circle, we must be familiar with some terms associated with these angles.



The part of a circle between any two points on the circumference is called an *arc*. An arc is named by its end points and any other point on it. Two points on the circumference denote 2 arcs, the longer one is the *major arc*, and the shorter one is the *minor arc*. The diagram shows major arc AB, or arc ACB, and minor arc AB, or arc ADB.

An angle with its vertex on the circumference and having chords for arms is called an *inscribed angle*. In the diagram, ∠ACB is an inscribed angle.

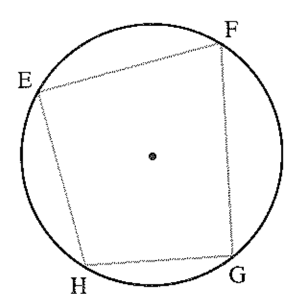
An angle with its vertex at the centre of the circle and having radii for arms is called a *sector angle*. In the diagram, ∠AOB is a sector angle.

Because ∠ACB and ∠AOB are formed by joining their vertices to the end points of the arc ADB, this arc ADB is said to *subtend* both ∠ACB and ∠AOB.

In a similar way, chord AB subtends both ∠ACB and ∠AOB.

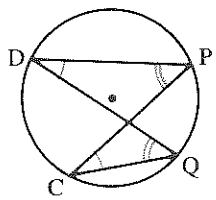
When a quadrilateral is inscribed in a circle; that is, its vertices lie on the circumference of the circle, it is called a *cyclic quadrilateral*.

Quadrilateral EFGH is a cyclic quadrilateral.

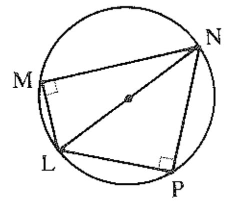


You may have discovered certain properties of the angles in a circle, from the previous *INVESTIGATE*. These properties are listed below.

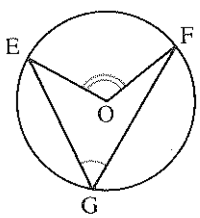
Inscribed angles subtended by the same arc are equal.
∠PDQ = ∠PCQ;
∠DPC = ∠DQC



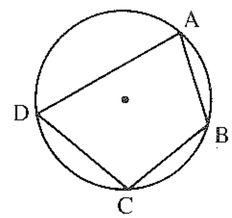
Inscribed angles subtended by a semicircle are 90°.
∠LMN = ∠LPN = 90°



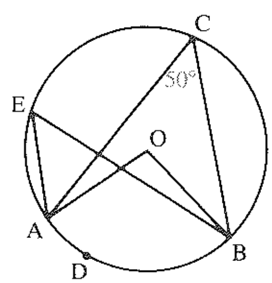
The measure of a sector angle is twice the measure of an inscribed angle subtended by the same arc.
∠EOF = 2∠EGF



In a cyclic quadrilateral, the opposite angles are supplementary.
∠ABC + ∠ADC = 180°
∠BAD + ∠BCD = 180°



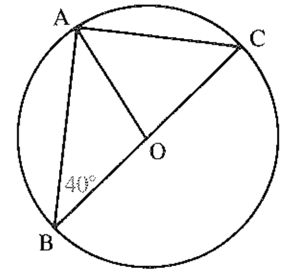
Example 1. O is the centre of the circle. Find the measure of each angle.
a) ∠AEB
b) ∠AOB



Solution. a) Since ∠AEB and ∠ACB are subtended by arc ADB,
∠AEB = ∠ACB
= 50°

b) Since ∠AOB is the sector angle subtended by the same arc as the inscribed angle ∠ACB,
∠AOB = 2∠ACB
= 2(50°)
= 100°

Example 2. BOC is a diameter of the circle.
Find the measure of each angle.
a) $\angle ACB$ b) $\angle OAB$

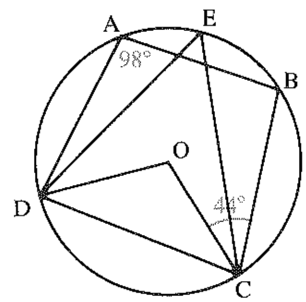


Solution. a) Since $\angle BAC$ is subtended by a semicircle,
$\angle BAC = 90°$
Since the sum of the angles in $\triangle BAC$ is 180°,

$$\begin{aligned} \angle BAC + \angle ACB + \angle CBA &= 180° \\ 90° + \angle ACB + 40° &= 180° \\ \angle ACB + 130° &= 180° \\ \angle ACB &= 50° \end{aligned}$$

b) Since OA and OB are radii of the circle,
$OB = OA$
Hence, $\triangle OAB$ is isosceles with $\angle OAB = \angle OBA$.
Therefore, $\angle OAB = 40°$

Example 3. O is the centre of the circle.
Find the measure of each angle.
a) $\angle DCO$ b) $\angle DEC$



Solution. a) Since $\angle DCB$ and $\angle DAB$ are opposite angles in a cyclic quadrilateral,
$\angle DCB + \angle DAB = 180°$
But $\angle DCB = \angle DCO + \angle OCB$

$$\begin{aligned} \angle DCO + \angle OCB + \angle DAB &= 180° \\ \angle DCO + 44° + 98° &= 180° \\ \angle DCO + 142° &= 180° \\ \angle DCO &= 38° \end{aligned}$$

b) Since OD and OC are radii, $\triangle ODC$ is isosceles.
$\angle ODC = \angle DCO = 38°$
Since the sum of the angles in $\triangle ODC$ is 180°,

$$\begin{aligned} \angle DOC + \angle ODC + \angle DCO &= 180° \\ \angle DOC + 38° + 38° &= 180° \\ \angle DOC + 76° &= 180° \\ \angle DOC &= 104° \end{aligned}$$

Since inscribed $\angle DEC$ and sector $\angle DOC$ are subtended by the same arc,

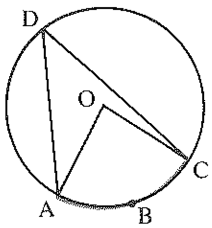
$$\begin{aligned} \angle DEC &= \frac{1}{2}\angle DOC \\ &= \frac{1}{2}(104°) \\ &= 52° \end{aligned}$$

EXERCISES 9-8

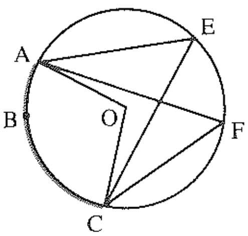
Ⓐ

1. For the arc ABC
   i) Name the inscribed angle(s). ii) Name the sector angle.

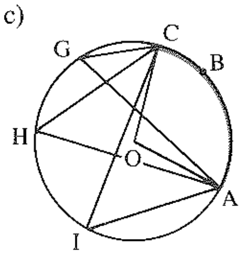
a)



b)

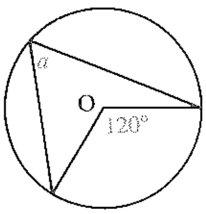


c)

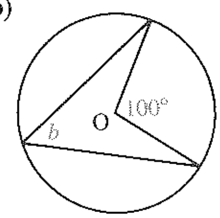


2. Find the angle measure indicated by each letter.

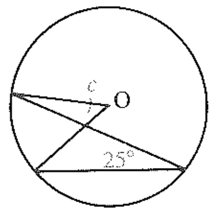
a)



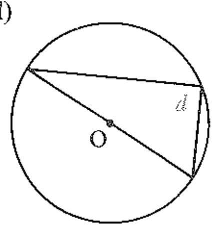
b)



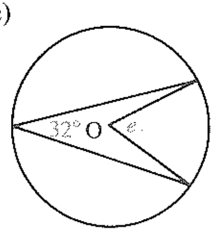
c)



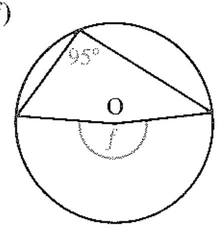
d)



e)

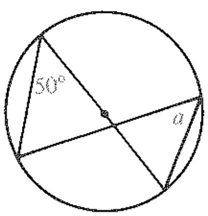


f)

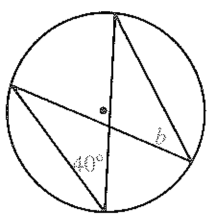


3. Find the angle measure indicated by each letter.

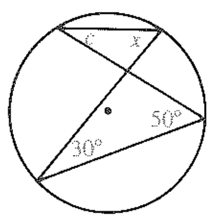
a)



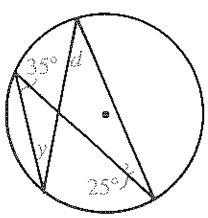
b)



c)



d)



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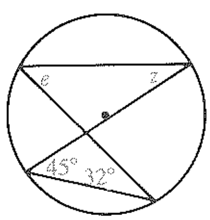
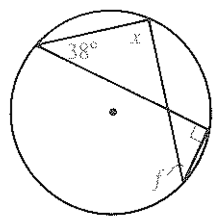
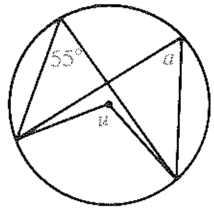


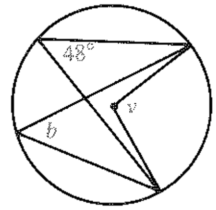
f)

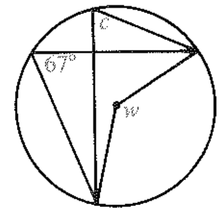


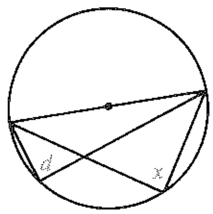
B

4. Find the angle measure indicated by each letter.

a


b)


c)


d)


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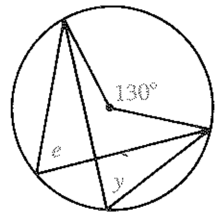
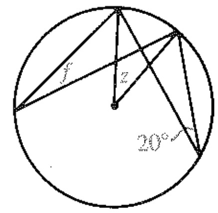
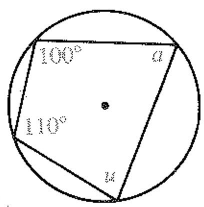
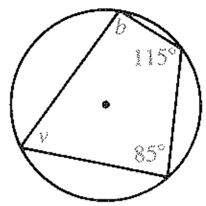
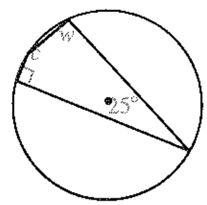


f)


5. Find the angle measure indicated by each letter.

a)


b)


c)


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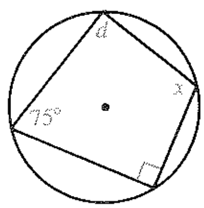

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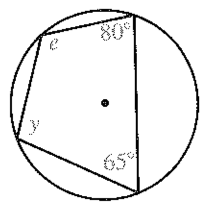

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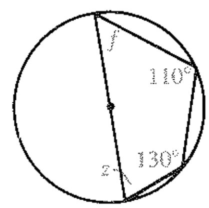
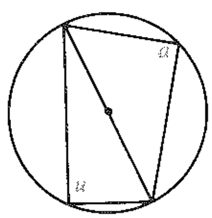
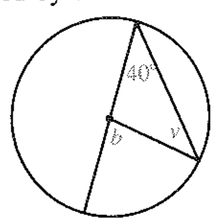
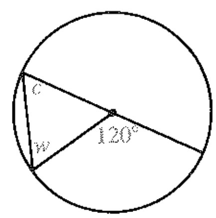


6. Find the angle measure indicated by each letter.

a)


b)


c)


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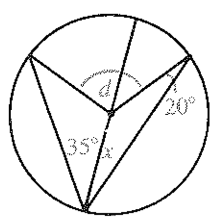

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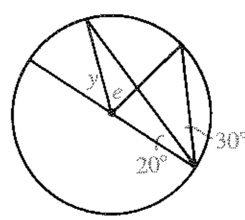

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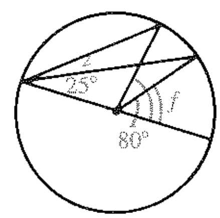
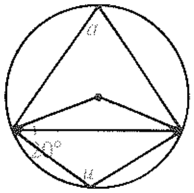
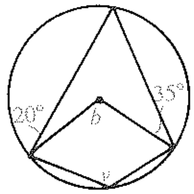


7. Find the angle measure indicated by each letter.

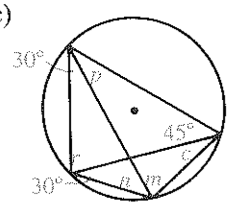
a)



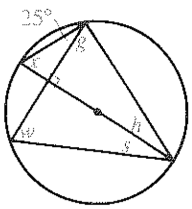
b)



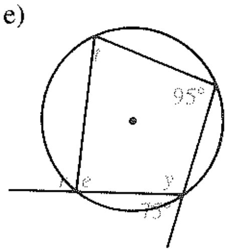
c)



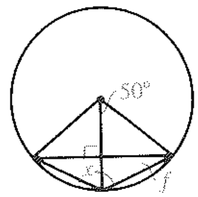
d)



e)

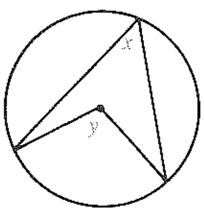


f)

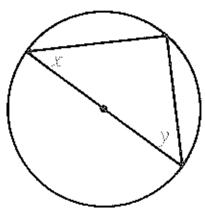


8. For each diagram
   i) If the value of x is known, how can the value of y be found?
   ii) Write an equation relating x and y.

a)



b)



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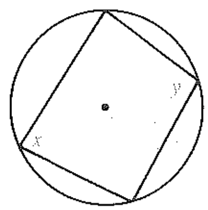

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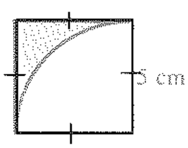
9. Find the area of each shaded region, to 1 decimal place.

a)



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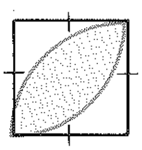
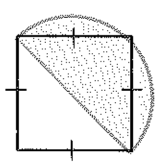


5 cm

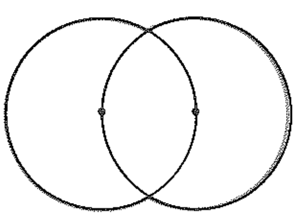
c)

5 cm



10. All the circles have a radius of 10 cm. The dots indicate the centres of the circles. Find the perimeter of each figure, to the nearest millimetre.

a)



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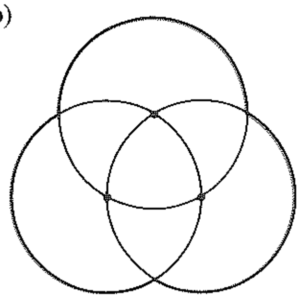
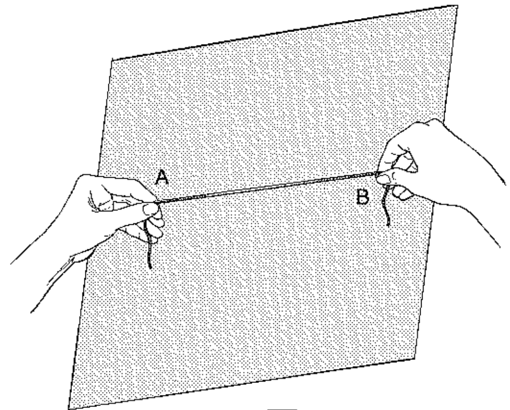


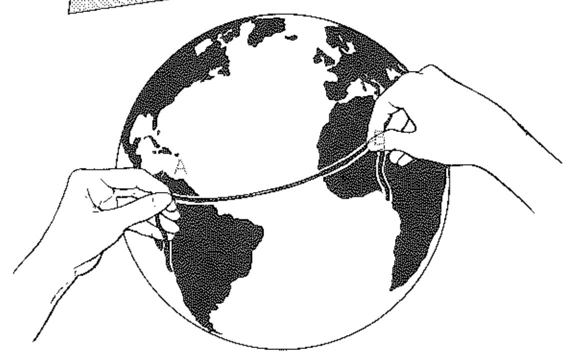
MATHEMATICS AROUND US

Geometry on the Earth's Surface

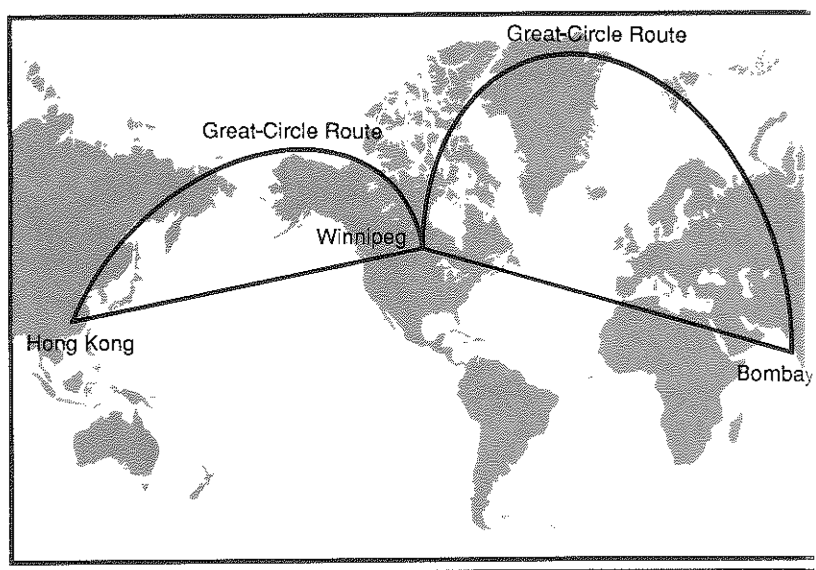
The geometry in this chapter is called plane geometry because the figures are all drawn on a plane. Geometry on the surface of the Earth is called *spherical geometry* because the figures are drawn on or visualized on a sphere. The shortest distance between two points can be represented by stretching a thread between them. On a plane, this is a straight line.



On a globe representing the Earth's surface, the shortest distance between two points is an arc of a *great circle* — the circle that is formed when a plane passes through the two points and the Earth's centre.

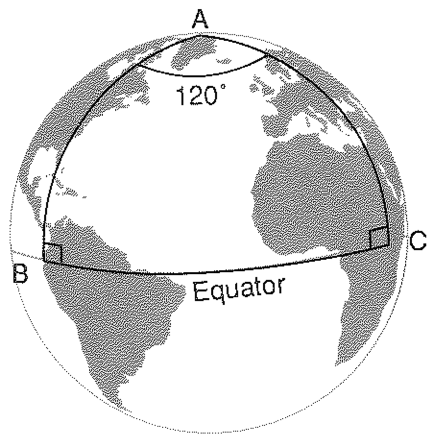


The shortest distance between two cities is an arc of a great circle, but it does not look that way on most maps. On the adjacent map the straight line joining Winnipeg and Bombay crosses the Atlantic Provinces and North Africa. This route, however, is actually much longer than the great-circle route which passes near the North Pole. Similarly, the shortest route from Winnipeg to Hong Kong is the great-circle route along the north coast of Alaska.



When a triangle is drawn on a plane, its three sides are line segments. The three sides of a *spherical triangle* are arcs of great circles.

On a globe representing the Earth, A is the North Pole, and B and C are points on the Equator. Spherical $\angle ABC$ is 90° because AB is a north/south line and BC is an east/west line. Similarly, spherical $\angle ACB$ is 90°. Spherical $\angle BAC$ is 120° because AC is the 40° E longitude line and AB is the 45° W longitude line. The sum of the angles of spherical $\triangle ABC$ is:
$90° + 90° + 120° = 300°$



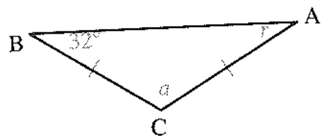
QUESTIONS

1. In the diagram above, if A and C are fixed and B moves along the Equator, how does the sum of the angles in spherical $\triangle ABC$ change?
2. On a globe, using tape and thread, show the great-circle routes from Winnipeg to Hong Kong and Bombay. Compare your routes with those shown on the maps.
3. Which routes do commercial aircraft fly? Why?
4. Which city on the globe is farthest from Winnipeg?
5. On a globe, with tape and thread, show the spherical triangles with vertices at the following cities. Measure their angles with a protractor. What is the sum of the angles in each spherical triangle?
   a) St. John's, Vancouver, Miami (Florida)
   b) Winnipeg, Cairo (Egypt), Rio de Janeiro (Brazil)
   c) Honolulu (Hawaii), Caracas (Venezuela), Nairobi (Kenya)
6. These statements are true for plane geometry. Are they true for spherical geometry? Explain.
   a) When two lines intersect, the opposite angles are equal.
   b) Parallel lines never meet.
   c) Each angle of an equilateral triangle measures 60°.
   d) The angles opposite the equal sides of an isosceles triangle are equal.

Review Exercises

1. Find the angle measure indicated by each letter.

a)



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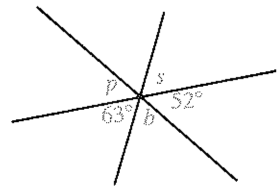

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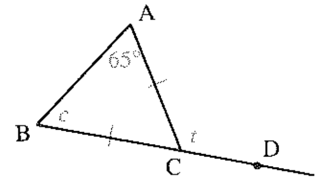

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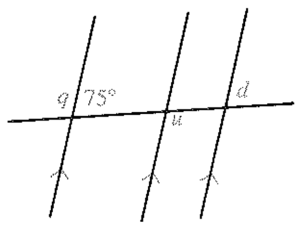

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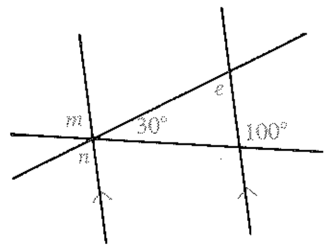

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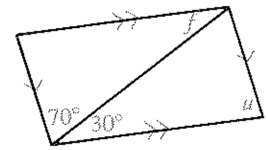
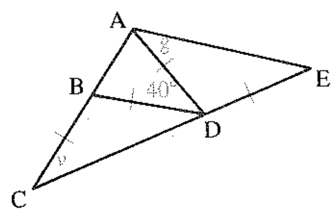


g)



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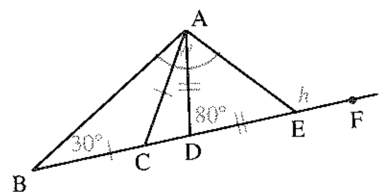
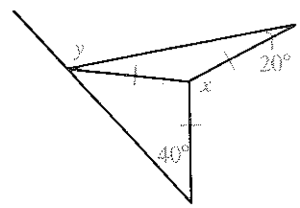
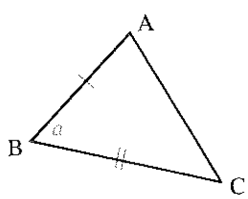


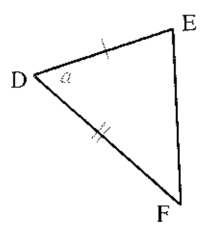
i)

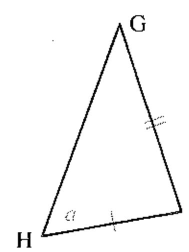


2. Find pairs of congruent triangles and state the condition for congruency.

a)







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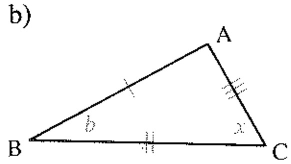
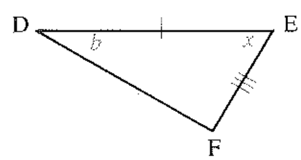
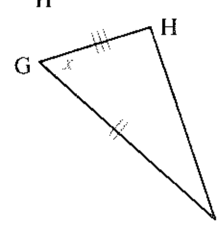
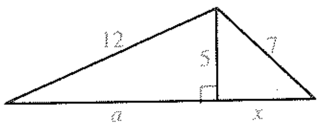




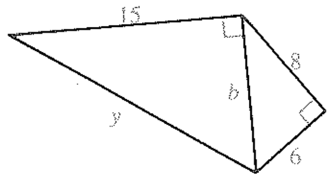


3. Find the length, to one decimal place, indicated by each letter.

a)



b)



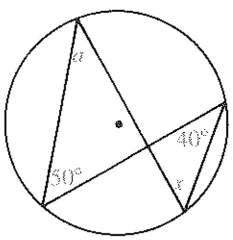
4. The size of a television screen is described by the length of its diagonal. Determine the size of a screen that is 34 cm by 40 cm.

5. Will a sheet of plywood 1200 mm wide fit into the rear of a station wagon if the door opening is 950 mm wide and 700 mm high?

6. The longest side of an isosceles right triangle is 50 cm. Find the length of the other two sides, to the nearest centimetre.

7. Find the angle measure indicated by each letter.

a)



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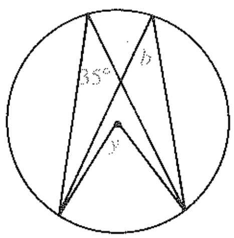
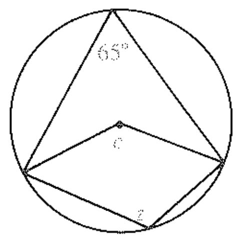
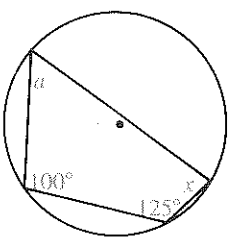


c)

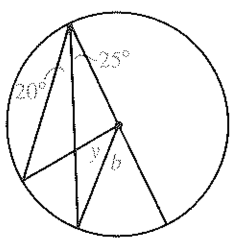


8. Find the angle measure indicated by each letter.

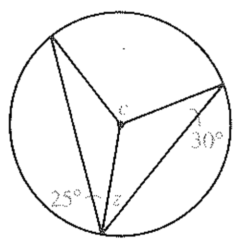
a)



b)



c)



9. Find each sum.
   a) the exterior angles of a triangle
   b) the interior angles of a quadrilateral
   c) the exterior angles of a quadrilateral

10. What is the length of each side of a square with diagonals of length 7 m? Give your answer to 2 decimal places.

11. What is the height of an equilateral triangle with sides 12 cm long? Give your answer to 1 decimal place.

Cumulative Review, Chapters 7-9

1. Simplify.
   a) $14m - 9n - 5m - 3n$
   b) $5a + 7b - 11a - 4b$
   c) $7x - 3 + 2x - 8 - 12x$
   d) $(3y^2 - 7y + 2) - (5y - y^2 + 9)$
   e) $(4x - 7) - (2x + 5) + (9x - 1)$
   f) $(8x - 9y + 4) + (3y - 5 - 6x) - (5x - 4y - 7)$

2. Evaluate, for each value of x.
   a) $5x^2 - 2x + 9$ b) $3(2x + 1) - x(x - 4) + 7$
   i) -2 ii) 5 iii) -1.5

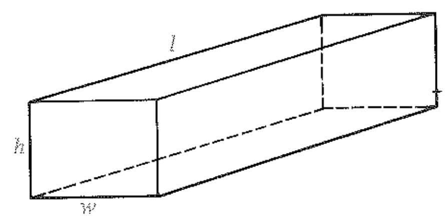
3. Simplify.
   a) $(3m)(-5m)$ b) $\left(-\frac{1}{2}x\right)^3$ c) $(-8a^3)(-6a^2)$
   d) $4x(2x^2 - 5x + 3)$ e) $(2xy^2)(-7x^2y^5)$ f) $-2m^2(5m^3 - 2m + 4n)$

4. Find each product.
   a) $(x - 4)(x + 11)$ b) $(2x - 7)(x - 3)$ c) $(4 + 3a)(5 + 9a)$
   d) $(3m - 5)(3m + 5)$ e) $x(2x + 3)(4x - 1)$ f) $-4a(7a + 2)(3a - 8)$

5. Factor.
   a) $6m^2 - 15m$ b) $2y^3 - 6y^2 + 8y$ c) $4x^2 - 49$
   d) $x^2 + 3x - 28$ e) $m^2 - 9m + 18$ f) $20 + a - a^2$

6. Factor.
   a) $y^2 + 16y + 39$ b) $2x^2 - 10x - 12$ c) $4a^3 - 100a$
   d) $-3x^3 + 15x^2 - 6x$ e) $3m^2 - \frac{3}{4}$ f) $y^3 - 15y^2 + 56y$

7. The surface area S of a rectangular prism with length l, width w, and height h, is given by this formula.
   $S = 2(lw + hw + lh)$
   Find the surface areas of prisms with these dimensions.



   a) length 20 cm; width 15 cm; height 8 cm
   b) length 17.5 cm; width 6.4 cm; height 3.2 cm

8. The number N of car accidents on a certain street in a week is given by this formula.
   $N = 2c^2 - 6c + 1$, where $c \geqslant 3$
   c is the number of cars, in thousands, that use the street each week. Find how many accidents there are in a week for each number of cars.
   a) 4000 b) 6000 c) 3000 d) 5000

9. Simplify.
   a) $\frac{-10x + 35}{-5}$ b) $\frac{8x^3 - 12x^2 + 20x}{4x}$ c) $\frac{4x^2y}{9xy^2} \times \frac{15xy}{16y^3}$
   d) $\frac{-17m^2n}{24mn} \times \frac{40m^4n^3}{51m^3n}$ e) $\frac{9x^5y^2}{16} \div \frac{3x^2y}{8}$ f) $\frac{-28a^2bc}{33ab^2} \div \frac{-21abc^3}{-44b^2c}$

10. Simplify.

a) $\dfrac{3x - 4}{3} + \dfrac{x + 5}{3}$ b) $\dfrac{2m + 7}{5} - \dfrac{3m + 1}{5}$

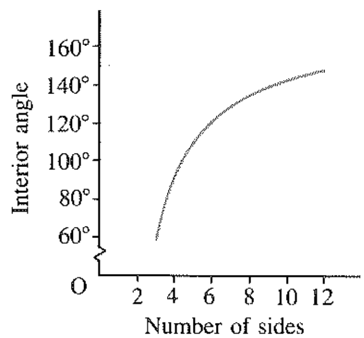
c) $\dfrac{3(2a - 5)}{4a} + \dfrac{2(a + 3)}{4a}$ d) $\dfrac{2(3x + 2)}{3} - \dfrac{5(2x - 7)}{4}$

11. Plot these points: A$(-3,1)$, B$(3,3)$, C$(5,-3)$, and D$(-4,-6)$. Join the points in order and name the figure drawn.

12. The graph shows the relation between the number of sides of a regular polygon and the size of its interior angles.

a) What is the size of the interior angle of:
i) an octagon ii) a dodecagon?

b) Name the polygon whose interior angle is:
i) 120° ii) 108°.



13. The sum of the first n natural numbers is given by this formula.

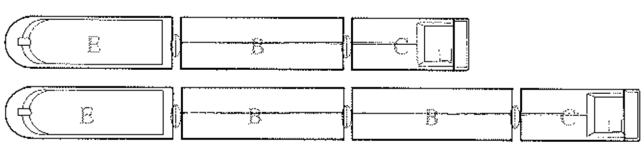
$$S = \frac{n(n + 1)}{2}$$

a) Make a table of values for S if $n = 1, 2, 3, 4, 5$, and 6.
b) Graph the relation defined by this table of values.
c) Why is 12 not a value for S in this relation?

14. Draw a graph of each relation.

a) $y = 2x + 1$ b) $x + 4y = 12$ c) $3x - 2y = 6$

15. When there is one boxcar on a train, we need 2 links to connect the engine and the caboose. When there are two boxcars, we need 3 links.



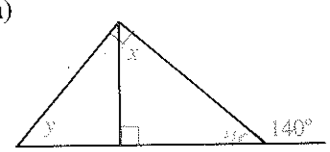
a) How many links are needed for 10 boxcars?
b) How many links are needed for 20 boxcars?
c) Let x represent the number of boxcars and let y represent the number of links in a train. Write an equation relating x and y.

16. The cost C dollars of a long distance telephone call is given by the formula, $C = 1.2m + 1.5$, where m is the time in minutes.

a) Graph this relation.
b) How much would each call cost? i) 4 min ii) 7 min
c) How long is a call which costs: i) \$5.10 ii) \$11.70?

17. Find the angle measure indicated by each letter.

a)



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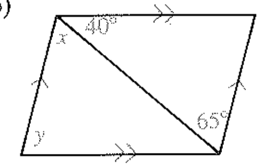

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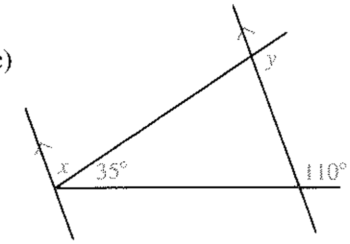
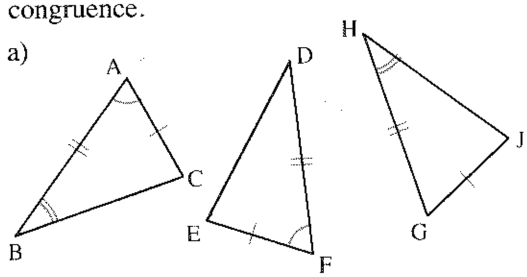


18. For each part, find a pair of congruent triangles and state the condition for their congruence.

a)



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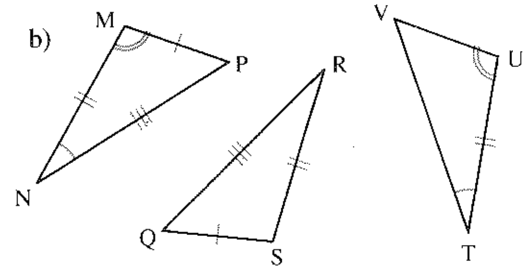
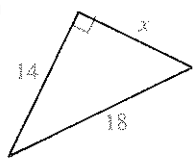


19. Find each value of x and y, to one decimal place.

a)



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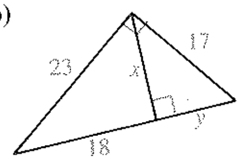

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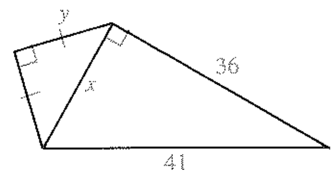
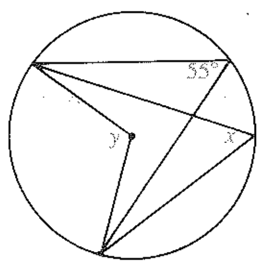


20. A conveyor belt 11.4 m in length is used to raise cartons of produce from one level to another, a distance of 3.7 m. How much floor space, measured in the direction of the conveyor, is required?

21. Find each value of x and y.

a)



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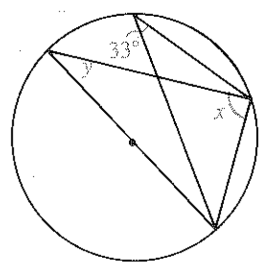

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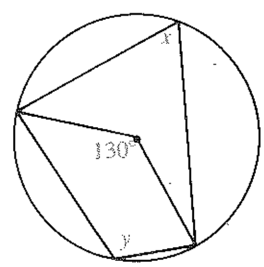

22. Safety procedures require the foot of a ladder to be no more than one-quarter of the length of the ladder away from the base of a building. Will a 10 m ladder safely reach a window 9.5 m high?

23. Draw each triangle.
   a) an isosceles obtuse triangle
   b) an equilateral triangle
   c) an isosceles right triangle
   d) a scalene acute triangle