5 Powers and Roots

Sunil's copy of this poster is 90 cm by 90 cm. He wants an enlargement that has twice the area. What will be the length of each side of the enlargement? (See Section 5-7, *Example 2*.)

5-1 THE MEANING OF EXPONENTS

Scientists use bacteria in medical research. The bacteria are grown in dishes called petri dishes, named after Julius Petri, a noted bacteriologist. The bacteria 'garden' is called a culture. By counting the bacteria in the culture at regular intervals of time, scientists can study how bacteria grow under controlled conditions.

The table shows a typical bacteria count every hour, starting with a bacteria count of 1000 at midnight.

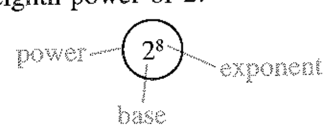
The number of bacteria doubles every hour.

Time	**Number of Bacteria**
midnight	1000
01:00	1000×2 or 2000
02:00	$1000 \times 2 \times 2$ or 4000
03:00	$1000 \times 2 \times 2 \times 2$ or 8000

If the pattern in the table is maintained, there would be at 08:00 a total of $1000 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ bacteria. This is a large number which can be written more simply using *exponents*.

The value, $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$, is written 2^8.

2^8 is a *power* with an exponent of 8 and a *base* of 2. We say that it is the eighth power of 2.



We say, "Two to the eighth."

The base is the number that is repeatedly multiplied. The exponent indicates how many of these numbers are multiplied.

The number of bacteria at 08:00 can be written as 1000×2^8 or 256 000 when expanded.

Example 1. Write each product as a power.

a) $(-9)(-9)(-9)(-9)$

b) $\left(\frac{7}{8}\right)\left(\frac{7}{8}\right)\left(\frac{7}{8}\right)\left(\frac{7}{8}\right)\left(\frac{7}{8}\right)$

c) $m \times m \times m$

d) $\left(\frac{2}{n}\right)\left(\frac{2}{n}\right)\left(\frac{2}{n}\right)\left(\frac{2}{n}\right)$

Solution.

a) $(-9)(-9)(-9)(-9) = (-9)^4$

b) $\left(\frac{7}{8}\right)\left(\frac{7}{8}\right)\left(\frac{7}{8}\right)\left(\frac{7}{8}\right)\left(\frac{7}{8}\right) = \left(\frac{7}{8}\right)^5$

c) $m \times m \times m = m^3$

d) $\left(\frac{2}{n}\right)\left(\frac{2}{n}\right)\left(\frac{2}{n}\right)\left(\frac{2}{n}\right) = \left(\frac{2}{n}\right)^4$

Example 2. If $n = 3$, evaluate: a) n^5 b) 5^n.

Solution.

a) If $n = 3$, $n^5 = 3^5$
$= (3)(3)(3)(3)(3)$
$= 243$

b) If $n = 3$, $5^n = 5^3$
$= (5)(5)(5)$
$= 125$

Example 3. Evaluate. a) $(-4)^3$ b) $(-4)^4$

Solution.

a) $(-4)^3 = (-4)(-4)(-4)$
$= -64$

b) $(-4)^4 = (-4)(-4)(-4)(-4)$
$= 256$

This example illustrates that

- a power with a negative base has a positive value when the exponent is even.
- a power with a negative base has a negative value when the exponent is odd.

The order of operations with exponents is the same as that for rational numbers. When evaluating expressions containing powers, evaluate the powers first unless brackets indicate otherwise.

Example 4. If $x = 5$ and $y = -2$ evaluate:

a) $x^2 - y^2$ b) $(x - y)^2$ c) $-3y^3$ d) $(-3y)^3$.

Solution. If $x = 5$ and $y = -2$

a) $x^2 - y^2 = 5^2 - (-2)^2$
$= 25 - 4$
$= 21$

b) $(x - y)^2 = [5 - (-2)]^2$
$= (7)^2$
$= 49$

c) $-3y^3 = -3(-2)^3$
$= -3(-8)$
$= 24$

d) $(-3y)^3 = [-3(-2)]^3$
$= (6)^3$
$= 216$

EXERCISES 5-1

(A)

1. Write each product as a power.
   a) $y \times y \times y \times y$
   b) $(-3)(-3)(-3)(-3)(-3)(-3)$
   c) $\frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$
   d) $\left(-\frac{3}{8}\right)\left(-\frac{3}{8}\right)\left(-\frac{3}{8}\right)$
   e) $(4a)(4a)(4a)(4a)(4a)$
   f) $2.9 \times 2.9 \times 2.9 \times 2.9$
   g) $m \times m \times m \times m$
   h) $(-6x)(-6x)(-6x)(-6x)(-6x)$
   i) $\pi \times \pi \times \pi \times \pi \times \pi \times \pi$
   j) $a \times a \times a \times a \times a \times a \times a$

2. Write each phrase as a power.
   a) six cubed
   b) seven to the eighth power
   c) nine to the fourth power
   d) twenty squared
   e) eleven to the fifth power
   f) five to the eleventh power
   g) four to the n^{th} power
   h) $2x$ to the tenth power

3. Evaluate.
   a) 4^3
   b) 3^4
   c) $(-2)^5$
   d) $(-5)^2$
   e) 10^4
   f) $\left(\frac{1}{4}\right)^2$
   g) $(0.2)^3$
   h) $(2.1)^2$
   i) $3(2^4)$
   j) $2^3\left(\frac{3}{4}\right)^2$
   k) $(3 \times 2)^4$
   l) $2(-3)^4$

4. Express each number as a power of 10.
   a) 1000
   b) 10 000
   c) 100
   d) 1 000 000
   e) 100 000
   f) 10
   g) 100 000 000
   h) 1 000 000 000 000

(B)

5. Using the pattern in the table on page 142, write an expression involving powers of 2 for the number of bacteria in the culture at each time.
   a) 01:00
   b) 02:00
   c) 06:00
   d) 10:00

6. Evaluate.
   a) $2^3 + 3^2$
   b) $3^2 + 4^2$
   c) $(3 + 4)^2$
   d) $5(-4)^3$
   e) $3^2 - 5^2$
   f) $(-4)^2 - 7^2$
   g) $\left(-\frac{3}{2}\right)^3 - \left(\frac{1}{4}\right)^3$
   h) $(-5)^3 - (-2)^5$

7. Evaluate $2x^2 - 3x + 5$ for each value of x.
   a) 4
   b) -1
   c) -2
   d) 10
   e) -5
   f) $\frac{1}{2}$
   g) $-\frac{1}{3}$
   h) 1.5
   i) -0.4
   j) 100

8. If $x = -3$ and $y = \frac{2}{3}$, evaluate each expression.
   a) x^2
   b) y^3
   c) $-5x^3$
   d) $-x^4$
   e) $(-x)^4$
   f) $x^2 + y^2$
   g) $x^2 - y^2$
   h) $(x + y)^3$
   i) $x^3 + y^3$
   j) $x^3 - y^3$
   k) $(x + y)^9$
   l) $(3x - y)^2$
   m) $5(x^2 - 2y)$
   n) $(3x + y)^3$
   o) $4x^2 - 7y^2$
   p) $4x^2 + y^2$

9. Evaluate each expression for $n = -3$.
   a) n^3 b) $n^2 - n^3$ c) $4n^2$ d) $(4n)^2$
   e) $-(n + 2)^3$ f) $(-n - 2)^8$ g) $(2 + n)^{15}$ h) $-(3n - 7)^4$
   i) $(-4n - 2n^2)^4$ j) $\left(\frac{n}{4}\right)^3$ k) $\left(\frac{2}{n} - 1\right)^5$ l) $(n - n^2)^3$

10. Arrange from greatest to least.
    a) 2^4, 3^2, 5^2, 2^3, 3^3
    b) $(-3)^4$, 4^3, 7^2, 2^5, 10^2
    c) $(1.2)^2$, $(1.1)^3$, $(1.05)^5$, $(1.15)^3$, $(1.3)^1$
    d) $(2.1)^4$, $(2.9)^3$, $(2.3)^2$, $(1.8)^7$, $(2.4)^5$
    e) $(0.3)^2$, $(0.2)^3$, $(0.2)^2$, $(0.3)^3$, $(0.4)^2$

Ⓒ

11. Use the information on page 142.
    a) If you knew how much time had elapsed since midnight, how could you find the number of bacteria in the culture?
    b) Write an expression for the number of bacteria in the culture n hours after midnight.
    c) At 01:30 there are about 2800 bacteria in the culture. Find approximately how many there would be:
       i) 1 h later ii) 2 h later iii) 1 h earlier.
    d) If a petri dish is half-covered by bacteria at midnight find when it will be completely covered.

12. Identify the greater number in each pair.
    a) 3^{22}, 3^{25} b) $3x^2$, $(3x)^2$ c) $(-5n)^3$, $5n^3$
    d) $(-2)^{16}$, $(-2)^{19}$ e) $(0.9)^{14}$, $(0.9)^{11}$ f) $\left(-\frac{3}{4}\right)^{10}$, $(-3.4)^7$

13. a) For what values of y is $y^2 < y$?
    b) For what values of x is $x^3 < x^4$?

14. Solve for n.
    a) $2^n = 8$ b) $3^n = 81$ c) $10^n = 1\,000\,000$
    d) $3(2^n) = 48$ e) $2(5^n) = 50$ f) $10(3^n) = 810$

15. Identify the greater number in each pair.
    a) 2^5, 5^3 b) 3^4, 4^3 c) 10^4, 2^{10}
    d) 6^4, 11^{11} e) 6^3, 3^6 f) 9^4, 3^8

16. Express the first number as a power of the second.
    a) 16, 4 b) 27, 3 c) 64, 2 d) 625, 5
    e) 16, -2 f) -243, -3 g) 343, 7 h) 256, -4
    i) 81, -3 j) 6561, 9 k) 7776, 6 l) 1.4641, 1.1

CALCULATOR POWER

Using a Scientific Calculator to Evaluate Powers

To evaluate $9^2 + (-8)^2$ . . .

. . . Linda pressed these keys on her calculator

$\boxed{9}\ \boxed{x^2}\ \boxed{+}\ \boxed{8}\ \boxed{+/-}\ \boxed{x^2}\ \boxed{=}$

. . . Dan pressed these keys

$\boxed{9}\ \boxed{x^2}\ \boxed{+}\ \boxed{-}\ \boxed{8}\ \boxed{x^2}\ \boxed{=}$

- What answers were obtained by Linda and Dan?
- What is the purpose of the $\boxed{+/-}$ key?
- Whose answer was correct?
- Explain where the error occurred in the incorrect keying sequence.

Evaluate each expression on your scientific calculator.

1. a) $6^2 + 12^2$ b) $5^2 - 9^2$ c) $(0.8^2) + (0.6^2)$ d) $(2.6^2) - (2.4^2)$
   e) $5^2 + (-7)^2$ f) $12^2 + (-5)^2$ g) $1.7(-9)^2$ h) $(2.3 - 12)^2$

To evaluate, on a scientific calculator, powers with exponents greater than 2 we use the $\boxed{y^x}$ (or $\boxed{a^x}$) key.

To evaluate 3^7, key in: $\boxed{3}\ \boxed{y^x}\ \boxed{7}\ \boxed{=}$ to display 2187

To evaluate $(-2)^9$, key in: $\boxed{2}\ \boxed{+/-}\ \boxed{y^x}\ \boxed{9}\ \boxed{=}$ to display -512

To evaluate $[5(-3)]^4$ . . .

. . . Linda pressed these keys on her calculator

$\boxed{5}\ \boxed{\times}\ \boxed{3}\ \boxed{+/-}\ \boxed{y^x}\ \boxed{4}\ \boxed{=}$

. . . Dan pressed these keys

$\boxed{5}\ \boxed{\times}\ \boxed{3}\ \boxed{+/-}\ \boxed{=}\ \boxed{y^x}\ \boxed{4}\ \boxed{=}$

- What answers did Linda and Dan obtain?
- Explain why their answers differ.
- Whose answer is correct?
- Explain where the error occurred in the incorrect sequence.

Evaluate each expression on your scientific calculator.

2. a) 17^3 b) 29^4 c) $(-3)^3$ d) $(-2)^7$

3. a) $0.3(5)^3$ b) $5.5(-4)^3$ c) $-7(3^4)$ d) $(8^3)(7^4)$
   e) $-6(-3)^7$ f) $(-9.1)^4(-2)^3$ g) $-(-2)^4(-3)^5$ h) $(-2)^3(3^2)(4^3)$

4. a) $2^9 - 3(5.7^4)$ b) $4(-3)^5 - 6(-2.8)^5$

5-2 EXPONENTS IN FORMULAS

In September 1979, an East German family crossed the heavily-guarded frontier into West Germany in a home-made hot air balloon. It was the first escape by this means. The balloon approximated the shape of a sphere with a diameter of about 22 m. What volume of air was in the balloon?

The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$, where r is the radius.

For the balloon, the radius is $\frac{1}{2}(22\text{ m})$ or 11 m.

$$V = \frac{4}{3}\pi r^3$$

$$= \frac{4}{3}\pi(11)^3 \quad \text{Use a calculator.}$$

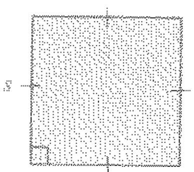
Key in: $\boxed{4}$ $\boxed{\div}$ $\boxed{3}$ $\boxed{\times}$ $\boxed{\pi}$ $\boxed{\times}$ $\boxed{1}$ $\boxed{1}$ $\boxed{y^x}$ $\boxed{3}$ $\boxed{=}$ to display 5575.2798

The volume of the balloon was about 5600 m^3.

The volume of air is usually expressed in kilolitres (kL).

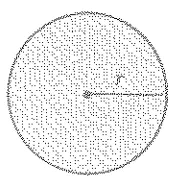
Since 1 m^3 = 1 kL, the volume of air in the balloon was about 5600 kL.

Other formulas which involve exponents are listed below.



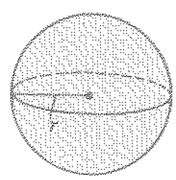
$A = s^2$

Area of a square



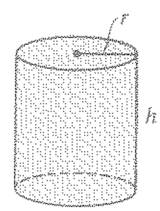
$A = \pi r^2$

Area of a circle



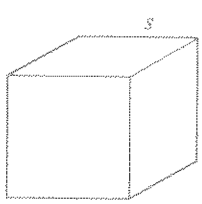
$A = 4\pi r^2$

Surface area of a sphere



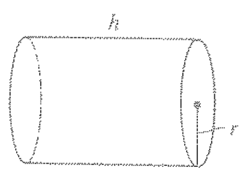
$A = 2\pi r^2 + 2\pi rh$

Surface area of a cylinder



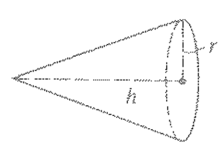
$V = s^3$

Volume of a cube



$V = \pi r^2 h$

Volume of a cylinder

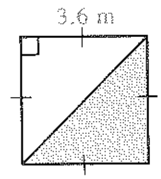


$V = \frac{1}{3}\pi r^2 h$

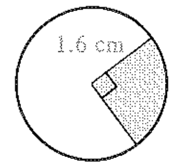
Volume of a cone

Example 1. Find the area, to one decimal place, of the shaded region of each figure.

a)



b)



Solution. a) Area of square, $A = s^2$

Shaded area of square, $\frac{1}{2}A = \frac{1}{2}s^2$

When $s = 3.6$, $\frac{1}{2}A = \frac{1}{2}(3.6)^2$

$= 6.48$

The shaded region of the square has an area of about 6.5 m^2.

b) Area of circle, $A = \pi r^2$

Shaded area of circle, $\frac{1}{4}A = \frac{1}{4}\pi r^2$

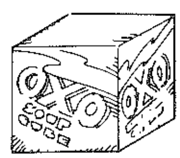
When $r = 1.6$, $\frac{1}{4}A = \frac{1}{4}\pi(1.6)^2$

$\doteq 2.01$

The shaded region of the circle has an area of about 2.0 cm^2.

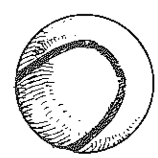
Example 2. Find the volume of each solid, to the nearest unit.

a)



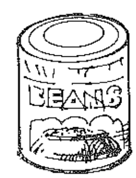
Cube of side 1.9 cm

b)



Sphere of diameter 9.4 cm

c)



Cylinder of height 11.0 cm, diameter 7.4 cm

Solution. a) Volume of cube, $V = s^3$

When $s = 1.9$, $V = (1.9)^3$

$= 6.859$

The volume of the cube is about 7 cm^3.

b) Volume of sphere, $V = \frac{4}{3}\pi r^3$

$r = \frac{1}{2}(9.4)$ or 4.7

$V = \frac{4}{3}\pi(4.7)^3$

$\doteq 434.89$

The volume of the sphere is about 435 cm^3.

If your calculator does not have a $\boxed{\pi}$ key, use $\pi = 3.14$.

c) Volume of cylinder, $V = \pi r^2 h$

$r = \frac{1}{2}(7.4)$ or 3.7 and $h = 11.0$

$V = \pi(3.7)^2(11.0)$
$\doteq 473.09$

The volume of the cylinder is about 473 cm^3.

Example 3. The distance d metres that an object falls from rest in t seconds is given by the formula $d \doteq 4.9t^2$. A pebble dropped from the top of a building takes 3.5 s to reach the ground. Find the height of the building.

Solution. Substitute $t = 3.5$ in the formula $d \doteq 4.9t^2$.

$d \doteq 4.9(3.5)^2$
$= 60.025$

The building is about 60 m high.

Example 4. A \$500 Canada Savings Bond pays 9% interest annually. This interest compounds (earns more interest) each year. The value V dollars of the bond after n years is given by this formula.

$V = 500(1.09)^n$

Find the value of the bond after: a) 2 years b) 5 years.

Solution. a) $V = 500(1.09)^n$

Substitute $n = 2$ into the formula.

$V = 500(1.09)^2$
$= 594.05$

After 2 years, the value of the bond is \$594.05.

b) $V = 500(1.09)^n$

Substitute $n = 5$ into the formula.

$V = 500(1.09)^5$
$\doteq 769.31$

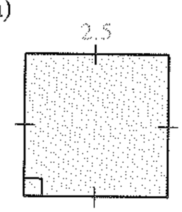
After 5 years, the value of the bond is \$769.31.

EXERCISES 5-2

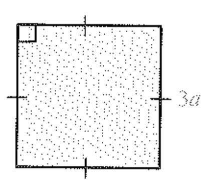
(A)

1. Express the area of each shaded region using exponents.

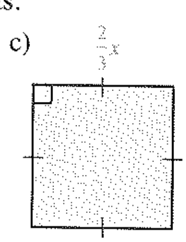
a)



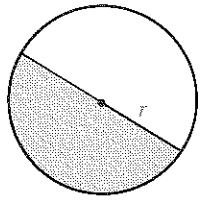
b)



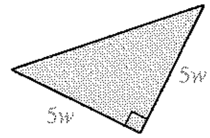
c)



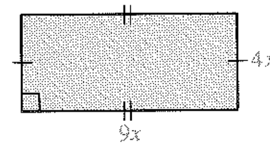
d)



e)

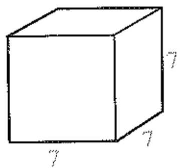


f)

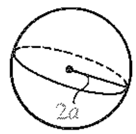


2. Express the volume of each solid using exponents.

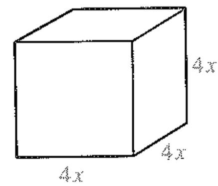
a)



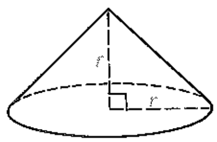
b)



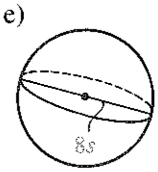
c)



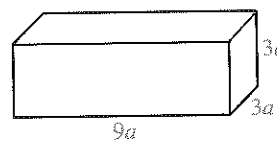
d)



e)



f)



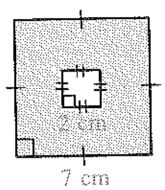
(B)

3. Find the area of each square with the given side length.
   a) 5 cm b) 9 m c) 1.5 cm d) 13.7 m
   e) 0.6 cm f) 2.6 m g) $4a$ units h) $5x$ units

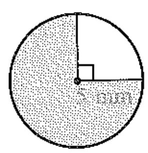
4. Find the volume of each cube having an edge length the same as that given in *Exercise 3*.

5. Find the area of each shaded region. Give the answers to the nearest square unit.

a)



b)



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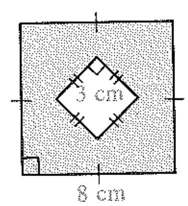
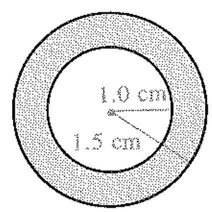
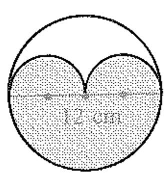


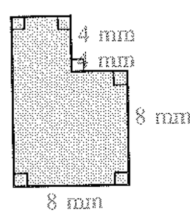
d)



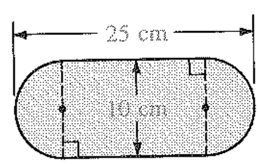
e)



f)



g)



6. A principal of \$200 in a savings account that pays 8% interest annually grows to \$200(1.08)^n in n years. What does the principal grow to in:
   a) 3 years b) 6 years c) 10 years?

7. A path 2 m wide is to enclose a circular lawn that has a 25 m radius. What will be the total cost of the material for the path if the cost per square metre is \$3.00?

8. A label just covers the curved surface of a soup tin with height 10 cm and diameter 7 cm. What is the area of the label?

9. A punch bowl is hemispherical and 50 cm in diameter. How many litres of punch can it hold? Give the answer to the nearest tenth of a litre.

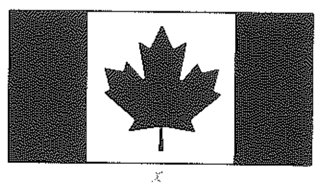
10. How many bouillon cubes with an edge length of 2 cm can be packed into a cubic box with an edge length of 0.5 m?

11. Find the volume of air contained in a spherical balloon with a radius of 12 cm. Give the answer to the nearest tenth of a litre.

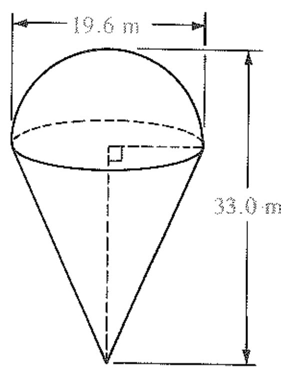
12. A car brakes and decelerates uniformly. The distance d metres that it travels in t seconds is given by this formula.
    $d = ut - 3.5t^2$
    u is the speed, in metres per second, just as the brakes are applied.
    Find how far the car travels while braking for 5 s, when it was travelling 25 m/s just as the brakes were applied.

Ⓒ

13. A class found by measuring that the relationship of the area A of the maple leaf on the Canadian flag to the flag's length x is $A \doteq 0.072x^2$.
    a) Find the area of the maple leaf on a flag of each length.
       i) 20 cm ii) 40 cm
       iii) 80 cm iv) 1.6 m
    b) Find the length of the flag that has a maple leaf with each area.
       i) 583.2 cm^2 ii) 1036.8 cm^2
       iii) 0.45 m^2 iv) 16.2 mm^2



14. The balloon that made the first successful crossing of the Atlantic Ocean was filled with helium. It had the shape of a hemisphere on a cone.
    The balloon was 33.0 m high and 19.6 m in diameter.
    Find the volume, to the nearest kilolitre, of helium that the balloon contained.



5-3 OPERATIONS WITH POWERS

Three operations involving powers are explained in this section.

- Multiplication of powers with the same base
- Division of powers with the same base
- Exponentiation of a power

All three operations are based on the definition of an exponent.

$x^n = x \times x \times x \ldots$ to n factors

Example 1. Simplify.

a) $5^3 \times 5^4$ b) $y^5 \times y^6$

Solution. a) $5^3 \times 5^4 = 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$
$= 5^7$

b) $y^5 \times y^6 = y \times y \times y \times y \times y \times y \times y \times y \times y \times y \times y$
$= y^{11}$

In each part of the example, the exponent of the simplified expression is the sum of the exponents in the original expression. This example illustrates the following rule for multiplying powers.

> To multiply powers with the same base, add the exponents.
> $x^m \times x^n = x^{m+n}$, $m > 0$, $n > 0$

Example 2. Simplify.

a) $7^8 \div 7^3$ b) $z^9 \div z^2$

Solution. a) $7^8 \div 7^3 = \dfrac{\cancel{7} \times \cancel{7} \times \cancel{7} \times 7 \times 7 \times 7 \times 7 \times 7}{\cancel{7} \times \cancel{7} \times \cancel{7}}$
$= 7^5$

b) $z^9 \div z^2 = \dfrac{\cancel{z} \times \cancel{z} \times z \times z \times z \times z \times z \times z \times z}{\cancel{z} \times \cancel{z}}$
$= z^7$

In each part of the example, the exponent of the simplified expression is the difference between the exponents in the original expression.

> To divide powers with the same base, subtract the exponents.
> $x^m \div x^n = x^{m-n}$, $m > n$, $m > 0$, $n > 0$, $x \neq 0$

Example 3. Simplify.

a) $(3^3)^4$ b) $(y^4)^5$

Solution.

a) $(3^3)^4 = 3^3 \times 3^3 \times 3^3 \times 3^3$
$= 3^{3+3+3+3}$
$= 3^{12}$

b) $(y^4)^5 = y^4 \times y^4 \times y^4 \times y^4 \times y^4$
$= y^{4+4+4+4+4}$
$= y^{20}$

In each part of the example, the exponent of the simplified expression is the product of the exponents in the original expression.

> To raise a power to an exponent, multiply the exponents.
> $(x^m)^n = x^{mn}$, $m > 0$, $n > 0$, $x \neq 0$

Example 4. Simplify.

a) $3a^4 \times 2a^3$ b) $\dfrac{12n^5}{6n^2}$ c) $\dfrac{5(b^2)^3 \times (3b^4)^2}{2b^3}$

Solution.

a) $3a^4 \times 2a^3 = 3 \times 2 \times a^{4+3}$
$= 6a^7$

b) $\dfrac{12n^5}{6n^2} = 2n^{5-2}$
$= 2n^3$

c) $$\frac{5(b^2)^3 \times (3b^4)^2}{2b^3} = \frac{5 \times 3^2 \times b^{2\times 3} \times b^{4\times 2}}{2b^3}$$
$$= \frac{45 \times b^6 \times b^8}{2b^3}$$
$$= \frac{45b^{6+8-3}}{2}$$
$$= \frac{45b^{11}}{2}$$

EXERCISES 5-3

(A)

1. Write each product as a power.
   a) $3^4 \times 3^6$ b) $7^4 \times 7^7$ c) $(-5)^{16}(-5)^9$
   d) $(2.1)^5(2.1)^{11}$ e) $(-8)^2(-8)^3(-8)$ f) $(-1.7)^4(-1.7)^2(-1.7)$
   g) $\left(\frac{2}{5}\right)^8\left(\frac{2}{5}\right)^{14}$ h) $\left(\frac{3}{11}\right)^{21}\left(\frac{3}{11}\right)^{15}$ i) $\left(-\frac{5}{4}\right)\left(-\frac{5}{4}\right)^6\left(-\frac{5}{4}\right)^7$

2. Simplify.
   a) x^7x^4 b) k^3k^9 c) n^6n^{17} d) $s^4s^5s^2$
   e) $v^{12}v^5v$ f) y^7yy^2 g) $(-a)^4(-a)^6$ h) $(-c)^7(-c)$

3. Simplify.

a) $3^8 \div 3^3$ b) $2^{16} \div 2^7$ c) $m^{20} \div m^5$ d) $\frac{s^{18}}{s^6}$

e) $\frac{14z^{12}}{-2z^4}$ f) $\frac{24r^{24}}{8r^8}$ g) $\frac{6^8}{6^2}$ h) $\frac{(-2)^7}{(-2)^3}$

4. Simplify.

a) $\frac{2^3 \times 2^5}{2^6}$ b) $\frac{3 \times 3^7}{3^2 \times 3^2}$ c) $\frac{m^4 \times m^3}{m^2}$

d) $\frac{b^4 \times b}{b^2}$ e) $\frac{(-a)^5(-a)}{(-a)^2}$ f) $\frac{x^{12} \times x^6}{x^5 \times x^4}$

g) $\frac{c^8 \times c^6}{c^2 \times c^9}$ h) $\frac{(-5)^{41} \times (-5)^{19}}{(-5)^{50}}$ i) $\frac{7^{14}}{7^3 \times 7^4}$

5. Simplify.

a) $(m^4)^5$ b) $[(-t)^3]^5$ c) $(a^7)^7$ d) $(2^3)^4$

e) $(12^5)^7$ f) $(10^2)^6$ g) $[(-5)^4]^3$ h) $(z^9)^3$

B

6. Simplify.

a) $3a^2 \times 5a^3$ b) $2m^3 \times 9m^5$ c) $4x^4 \times 9x^9$

d) $6y^5(-3y^7)$ e) $5(3)^8 \times 6(3)^4$ f) $8(-7)^4 \times 4(-7)^{11}$

g) $3x \times 19x^{10}$ h) $2p^5 \times 5p^2 \times 3p^3$ i) $4s^5 \times 7s^{10} \times 3s$

7. Simplify.

a) $\frac{20d^5}{4d^2}$ b) $\frac{36a^{12}}{-4a^3}$ c) $\frac{-42z^3}{-7z^2}$

d) $15m^9 \div 5m^3$ e) $-32x^{12} \div 8x^4$ f) $50a^{20} \div 20a^5$

g) $12(2)^7 \div 4(2)^3$ h) $-24(3)^{18} \div 4(3)^6$ i) $\frac{4n^{12} \times 5n^3}{10n^6}$

j) $\frac{3c^6 \times 2c}{4c^2}$ k) $\frac{18m^{14} \times 5m^7}{10m^3}$ l) $\frac{-9a^7 \times (-8)a^9}{-18a^8}$

8. Simplify.

a) $(6m^2)^3$ b) $(4x^5)^2$ c) $(13a^7)^4$ d) $(-3p^2)^8$

e) $(-6c^4)^6$ f) $(-3x^5)^2$ g) $\frac{(2k^2)^2}{k^3}$ h) $\frac{(3n^4)^3}{(2n)^2}$

i) $\frac{(4a^5)^3(4a^6)^2}{4a^4}$ j) $\frac{-(-2n^4)^2(-n^2)^4}{(3n)^3}$ k) $\frac{(0.5x^2)^3(-2x^3)^2}{(-0.5x)^2}$ l) $\frac{(10m^3)^4(0.1m^2)^4}{(100m^2)(0.01m)^2}$

9. Astronomers estimate that there are about 10^{11} galaxies in the universe, and that each galaxy contains about 10^{11} stars. About how many stars are there in the universe?

C

10. Simplify.

a) $(a^2b^4)^3$ b) $(3x^2y^3z)^4$ c) $(-2mn^3p^4)^3$ d) $\frac{(-36m^2n^3)^2}{(12m^3n^2)^3}$ e) $\frac{(4ab^2c^3)^3}{(-3a^3b^2c)^4}$

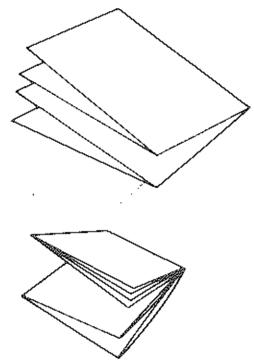
11. Evaluate each expression for: i) $a = 2$ ii) $a = 0.5$ iii) $a = -5$.

a) $a \times a^3$ b) $a^2 \times a^4$ c) $a^9 \div a^7$ d) $\dfrac{a^3 \times a^5}{a^4}$

e) $\dfrac{15a^3}{5a}$ f) $\dfrac{(-6a^2)(8a^3)}{(4a^2)^2}$ g) $(a^2)^3$ h) $(3a^2)^2$

12. a) Make a table of powers of 2 up to 2^{24}.
b) Use your table of powers of 2 to evaluate each expression.
i) 16×64 ii) 32×512 iii) $65\,536 \div 2048$
iv) $\dfrac{128 \times 4096}{32}$ v) 64^3 vi) 4^5

13. Fold a piece of paper in half (giving 2 layers of paper). Fold it in half again (giving 4 layers). Fold it in half again (giving 8 layers), and continue folding it in half in this manner.
a) Suppose you could do this 20 times.
i) How many layers of paper would there be?
ii) About how thick would the resulting wad of paper be?
b) Find out how many times you can actually fold a piece of paper in this way.



INVESTIGATE

There are two possible meanings for an expression such as 2^{3^2}, depending on the order in which the exponents are calculated.
2^{3^2} might mean $(2^3)^2$ or it might mean $2^{(3^2)}$.

1. Evaluate the expressions $(2^3)^2$ and $2^{(3^2)}$. Are they the same?
2. Mathematicians have agreed that, in an expression such as 2^{3^2}, the exponents should be evaluated starting at the top. Using this convention, find the value of each expression.
a) 2^{3^2} b) 3^{2^2} c) 2^{2^3} d) 5^{2^2}
3. Evaluate each expression.
a) 2^2 b) 2^{2^2} c) $2^{2^{2^2}}$
4. Mathematicians have calculated that $2^{2^{2^{2^2}}}$ is a number with 19 729 digits. Estimate the number of pages of this book that would be needed to print the number $2^{2^{2^{2^2}}}$.

It has been estimated that the number $2^{2^{2^{2^{2^2}}}}$ is so large that it can never be calculated, because the answer would require the age of the universe in computer time, and the space of the universe to hold the printout.

5-4 THE EXPONENT LAWS

Consider the statement, $\dfrac{x^m}{x^n} = x^{m-n}$, $x \neq 0$.

We have shown that this is true if m and n are positive integers and $m > n$. What happens to this equation if the restriction $m > n$ does not hold? That is,

Suppose $m = n$

For example, if $m = 3$ and $n = 3$ and $x \neq 0$,

then $\dfrac{x^3}{x^3} = \dfrac{(x)(x)(x)}{(x)(x)(x)} = 1$ also $\dfrac{x^3}{x^3} = x^{3-3} = x^0$

Since x^m is defined only for positive values of m, the expression x^0 has not yet been defined.

However, if we define x^0 to be 1, then $\dfrac{x^3}{x^3} = 1 = x^{3-3}$, for $x \neq 0$.

That is, if we define x^0 to be 1 for $x \neq 0$, then the exponent law $\dfrac{x^m}{x^n} = x^{m-n}$, $x \neq 0$, is true for $m > n$ and $m = n$.

For any number x, where $x \neq 0$, $x^0 = 1$

Suppose $m < n$

For example, if $m = 3$ and $n = 7$ and $x \neq 0$,

then $\dfrac{x^3}{x^7} = \dfrac{x \times x \times x}{x \times x \times x \times x \times x \times x \times x} = \dfrac{1}{x^4}$ also $\dfrac{x^3}{x^7} = x^{3-7} = x^{-4}$

Since x^m is defined only for non-negative values of m, the expression x^{-4} has not yet been defined.

However, if we define x^{-4} to mean $\dfrac{1}{x^4}$, then $\dfrac{x^3}{x^7} = x^{-4}$, for $x \neq 0$.

That is, if we define x^{-n} to mean $\dfrac{1}{x^n}$ for $x \neq 0$, then the exponent law, $\dfrac{x^m}{x^n} = x^{m-n}$, $x \neq 0$, is true for all integral values of m and n.

For any number x and any integer n, x^{-n} is defined to be the reciprocal of x^n. $x^{-n} = \dfrac{1}{x^n}$, $x \neq 0$

These two statements, along with the 3 statements from the previous section, comprise the *exponent laws*.

> **Exponent Laws**
> For all integers m and n
> - $x^m \times x^n = x^{m+n}$
> - $x^m \div x^n = x^{m-n}, x \neq 0$
> - $(x^m)^n = x^{mn}$
> - $x^0 = 1, x \neq 0$
> - $x^{-m} = \dfrac{1}{x^m}, x \neq 0$

The pattern of this table is a further illustration of the relationship between positive and negative exponents.

Reducing exponents by 1,

$$\begin{aligned}
2^4 &= 2 \times 2 \times 2 \times 2 = 16\\
2^3 &= 2 \times 2 \times 2 = 8\\
2^2 &= 2 \times 2 = 4\\
2^1 &= 2 = 2\\
2^0 &= 1 = 1\\
2^{-1} &= \frac{1}{2} = \frac{1}{2}\\
2^{-2} &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}\\
2^{-3} &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\\
2^{-4} &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}
\end{aligned}$$

divides answers by 2.

Example 1. Evaluate.

a) 5^{-2} b) $\left(\frac{1}{4}\right)^{-3}$ c) $(3^0 - 3^{-1})^{-2}$

Solution.

a) $5^{-2} = \dfrac{1}{5^2}$
$= \dfrac{1}{25}$

b) $\left(\dfrac{1}{4}\right)^{-3} = \dfrac{1}{\left(\frac{1}{4}\right)^3}$
$= \dfrac{1}{\frac{1}{64}}$
$= 64$

c) $$(3^0 - 3^{-1})^{-2} = \left(1 - \frac{1}{3}\right)^{-2} = \left(\frac{2}{3}\right)^{-2} = \frac{1}{\left(\frac{2}{3}\right)^2} = \frac{1}{\frac{4}{9}} = \frac{9}{4}$$

or

$$(3^0 - 3^{-1})^{-2} = \left(1 - \frac{1}{3}\right)^{-2} = \left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

Example 2. Evaluate.

a) $4^{-3} \times 4^2 \times 4^{-1}$ b) $(-2)^{-4} \div (-2)^{-1}$ c) $(3^{-1})^{-2}$

Solution. a) $$4^{-3} \times 4^2 \times 4^{-1} = 4^{-3+2-1} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}$$

b) $$(-2)^{-4} \div (-2)^{-1} = (-2)^{-4-(-1)} = (-2)^{-3} = \frac{1}{(-2)^3} = -\frac{1}{8}$$

c) $$(3^{-1})^{-2} = 3^{(-1)(-2)} = 3^2 = 9$$

Example 3. Simplify.

a) $(x^{-2})(x^5)(x^0)$ b) $x^3 \div x^{-5}$ c) $(x^2)^{-3}$

Solution. a) $$(x^{-2})(x^5)(x^0) = x^{-2+5+0} = x^3$$

b) $$x^3 \div x^{-5} = x^{3-(-5)} = x^8$$

c) $$(x^2)^{-3} = x^{2(-3)} = x^{-6}$$

Example 4. Evaluate each expression for $a = -2$ and $b = 3$.

a) $a^{-2} - b^{-1}$ b) $3a^{-1} + 2b^{-2}$

Solution. If $a = -2$ and $b = 3$

a) $$\begin{aligned} a^{-2} - b^{-1} &= (-2)^{-2} - (3)^{-1} \\ &= \frac{1}{(-2)^2} - \frac{1}{3} \\ &= \frac{1}{4} - \frac{1}{3} \\ &= \frac{3}{12} - \frac{4}{12} \\ &= -\frac{1}{12} \end{aligned}$$

b) $$\begin{aligned} 3a^{-1} + 2b^{-2} &= 3(-2)^{-1} + 2(3)^{-2} \\ &= \frac{3}{-2} + \frac{2}{3^2} \\ &= -\frac{3}{2} + \frac{2}{9} \\ &= -\frac{27}{18} + \frac{4}{18} \\ &= -\frac{23}{18} \end{aligned}$$

EXERCISES 5-4

(A)

1. Evaluate.

a) 2^{-1} b) 5^{-1} c) 3^{-2} d) 2^{-3} e) 5^{-3} f) 10^{-2}

g) 12^{-3} h) 10^{-4} i) $\left(\frac{1}{2}\right)^0$ j) $\left(\frac{1}{4}\right)^{-2}$ k) 10^{-5} l) $\frac{1}{5^{-1}}$

m) $\frac{1}{2^{-5}}$ n) $\frac{3}{4^{-2}}$ o) $\left(\frac{3}{4}\right)^{-2}$ p) $\left(\frac{1}{10}\right)^{-1}$ q) $(0.1)^{-3}$ r) $(0.5)^{-2}$

2. Simplify.

a) $10^3 \times 10^{-5}$ b) $10^{-4} \div 10^{-3}$

c) $(10^{-4})^2 \times 10^{-1}$ d) $10^0 \times 10^8$

e) $10^4 \div 10^{-5} \times 10^6$ f) $10^4 \times 10^{-5} \div 10^6$

g) $(10^{-5})^3(10^5)^3$ h) $(10^4 \times 10^6)^0 \div 10^{-1}$

i) $(10^{-3})^4(10^{-4})^3 \div (10^{-2})^{-1}$ j) $10^{-5} \div 10^{-3} \times (10^4)^{-2}$

(B)

3. Write each expression as a power.

a) $5^4 \times 5^7$ b) $2^{-5} \times 2^{11}$ c) $3^4 \div 3^{-11}$

d) $7^{-8} \div 7^2$ e) $11^{-13} \times 11^{20}$ f) $(-5)^{-11} \div (-5)^{19}$

g) $6^{-8} \times 6^{-15}$ h) $(-9)^4 \div (-9)^4$ i) $19^{-7} \div 19^{12}$

4. Simplify.
   a) $x^{-9} \times x^{-4}$
   b) $p^{-7} \div p^{2}$
   c) $w^{-13} \times w^{8}$
   d) $y^{5} \times y^{-9}$
   e) $x^{-5} \div x^{13}$
   f) $a^{7} \div a^{-4}$
   g) $m^{-14} \div m^{-5}$
   h) $s^{-5} \times s^{17}$
   i) $t^{-9} \div t^{-17}$

5. Evaluate.
   a) $3^{2} + 3^{-2}$
   b) $3^{2} - 3^{-2}$
   c) $3^{-2} - 3^{2}$
   d) $3^{2} \times 3^{-2}$
   e) $3^{2} \div 3^{-2}$
   f) $3^{-2} \div 3^{2}$

6. Evaluate.
   a) $2^{3} - 2^{-1}$
   b) $5^{2} + 5^{-1}$
   c) $7^{-2} - 7$
   d) $(2 \times 3)^{-2}$
   e) $4^{2} + 4^{0}$
   f) $3^{-1} + 3^{-2}$
   g) $6^{2} + 6^{0} + 6^{-2}$
   h) $(2^{2} - 1)^{-2}$
   i) $3^{-2} - 2^{-4}$

7. Evaluate.
   a) $(-2)^{3}$
   b) $(-2)^{-3}$
   c) $-(-2)^{-3}$
   d) $(-5)^{0}$
   e) $-(5^{0})$
   f) $(6 - 4)^{-3}$
   g) $(5 - 8)^{-1}$
   h) $\left(\frac{1}{4} - \frac{1}{4^{2}}\right)^{-2}$
   i) $[(-3)^{-2} + (-3)^{-1}]^{-1}$

8. Express as powers of 2, and then arrange in order from greatest to least.
   $\frac{1}{32}, 16, 128, \frac{1}{64}, 1, \frac{1}{2}$

9. Express as powers of 3, and then arrange in order from least to greatest.
   $\frac{1}{9}, \frac{1}{243}, \frac{1}{81}, 27, \frac{1}{729}, 1$

10. Express as powers with positive exponents.
   a) 49
   b) $\frac{1}{100}$
   c) $\frac{1}{343}$
   d) $\frac{1}{-32}$
   e) $\frac{1}{1\,000\,000}$
   f) 0.25
   g) 0.001
   h) 0.125

11. Express as powers with negative exponents other than -1.
   a) $\frac{1}{121}$
   b) $\frac{1}{169}$
   c) 0.01
   d) 0.1
   e) 0.000 01
   f) 0.008
   g) 0.0081
   h) $\frac{1}{1728}$

12. Simplify.
   a) $a^{-3} \div a^{6} \div a^{-8}$
   b) $y^{-5} \div y^{9} \times y^{4}$
   c) $(-3)^{-6} \times (-3)^{4} \times (-3)^{3}$
   d) $2^{3} \times 2^{-7} \div 2^{-5}$
   e) $m^{-6} \div m^{-2} \times m^{-9}$
   f) $x^{-4} \times x^{-8} \div x^{-7}$
   g) $p^{11} \div p^{15} \div p^{-9}$
   h) $(-7)^{12} \times (-7)^{-8} \div (-7)^{17}$
   i) $11^{5} \div 11^{10} \div 11^{-3}$
   j) $(0.5)^{2} \div (0.5)^{-2} \times (0.5)^{-5}$
   k) $(-0.1)^{4} \times 10^{6} \times (0.01)^{2}$
   l) $10^{-3} \div 100^{-2} \times (0.1)^{-1}$
   m) $(-2)^{-4} \times (-0.5)^{-5} \times 2^{-2}$
   n) $(0.25)^{8} \div 4^{-7} \times 2^{2}$

13. Simplify.
a) $(x^{-2})^3 \div (x^3)^2$ b) $(y^4)^2 \times (y^{-2})^3$ c) $(3^2)^5 \times (3^3)^2$
d) $(2^{-4})^2 \div (2^2)^6$ e) $(m^{-3})^4 \div (m^4)^{-3}$ f) $(8^3)^{-3} \times (8^2)^{-2}$
g) $(w^2)^{-7} \times (w^{-3})^{-4}$ h) $(5^{-3})^4 \div (5^2)^3$ i) $(x^2)^4 \times (x^{-4})^5$

14. Evaluate.
a) $5^{-1} \div 3^{-2}$ b) $(3^{-1} - 3^{-2})^{-1}$ c) $\left(\frac{1}{4}\right)^{-1} - \left(\frac{1}{3}\right)^{-2}$
d) $\left(\frac{1}{-2}\right)^{-3} + \left(\frac{1}{2}\right)^{-2}$ e) $\left(\frac{2}{3^{-1}}\right)^{-3}$ f) $(0.5)^{-3} + (0.5)^0$
g) $\dfrac{4}{4^{-1} + 4^0}$ h) $\dfrac{2^{-1}}{2^{-2} - 2^{-3}}$ i) $[47(5)^{-2}]^0$

15. Simplify.
a) $5n^{-4} \times 2n^{17}$ b) $12t^4 \div 3t^{-3}$ c) $60x^5 \div 12x^{-5}$
d) $16w^{-8} \div 4w^{-2}$ e) $7a^{-4} \times (-4a^{-2})$ f) $-12y^{-9} \times 6y^{17}$
g) $15s^{-15} \div 3s^5$ h) $-4m^{-7} \times (-3m^{-2})$ i) $18x^5 \div (-3x^8)$

16. Simplify.
a) $2m^{-3} \times 5m^{-4} \times 3m^{11}$ b) $6a^2 \div (-2)a^5 \times 4a^{-7}$
c) $24y^6 \div 3y^2 \div 2y^{-2}$ d) $45b^{-3} \div 5b^5 \times 3b^{-7}$
e) $-9m^{-7} \times 8m^{-2} \div (-6m^{-3})$ f) $-15y^{-4} \div 5y^8 \div 3y^{-12}$

17. Evaluate each expression for $a = -3$, $b = 2$, and $c = -1$.
a) a^{-1} b) $-a^{-1}$ c) $a^{-1} + b^{-1}$
d) $(a + b + c)^{-1}$ e) $a^{-1} + b^{-1} + c^{-1}$ f) a^b
g) $\left(\frac{a}{b - c}\right)^{-2}$ h) $\left(\frac{2a}{b + 4c}\right)^{-3}$ i) $\left(\frac{-3a}{-2b + c}\right)^{-2}$

18. Evaluate $3a^{-2} + b^c$ for these values of a, b, and c.
a) $a = 4, b = 3, c = 0$ b) $a = 3, b = 2, c = -1$
c) $a = -\frac{1}{2}, b = -2, c = 3$ d) $a = \frac{2}{3}, b = \frac{5}{4}, c = -1$

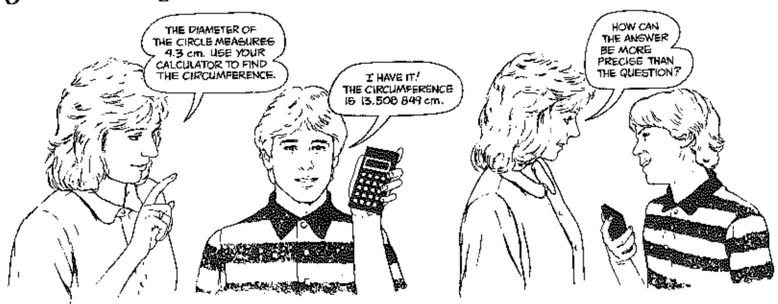
19. Evaluate each expression for: i) $x = 2$ ii) $x = -\frac{1}{2}$.
a) x^3 b) $(-x)^3$ c) $-x^3$ d) $-(-x)^3$
e) x^{-3} f) $(-x)^{-3}$ g) $-x^{-3}$ h) $-(-x)^{-3}$

C

20. Solve.
a) $5^x = 1$ b) $2^x = \frac{1}{2}$ c) $(-3)^x = \frac{1}{9}$ d) $x^{-3} = \frac{1}{125}$
e) $2^x = \frac{1}{32}$ f) $x^2 = \frac{1}{25}$ g) $4^{x-1} = \frac{1}{64}$ h) $10^{2-x} = 0.001$
i) $2^{-x-4} = \frac{1}{32}$ j) $243 = \left(\frac{1}{3}\right)^{x+4}$ k) $64 = (0.5)^{3-x}$ l) $9^{1+x} = 27$

CALCULATOR POWER

Significant Digits



When measuring the diameter of a circle, we may find that the diameter is 4.3 cm *to the nearest tenth of a centimetre*. This means that the actual diameter is between 4.25 cm and 4.35 cm. We say that the digits 4 and 3 in the measurement 4.3 cm are *significant digits*.

The student in the picture multiplied 4.3 by π to obtain the circumference. When he pressed the $\boxed{\pi}$ key, the calculator used the value 3.141 592 7 for π. Then the student obtained the result 13.508 849. A circumference of 13.508 849 cm represents a measurement that is correct to the nearest millionth of a centimetre. No wonder the teacher was surprised to find that the diameter of a circle known only to the nearest tenth of a centimetre was used to calculate a circumference to the nearest millionth of a centimetre!

When the student multiplied the diameter by π he should have realized that his diameter was known only to be between 4.25 cm and 4.35 cm. If he had used both of these values with his calculator, he would have found the following results for the circumference.

Using a diameter of 4.25 cm, the circumference is:

$$\begin{aligned} C &= \pi d \\ &= 3.141\ 592\ 7(4.25) \\ &= 13.351\ 769 \end{aligned}$$

Using a diameter of 4.35 cm, the circumference is:

$$\begin{aligned} C &= \pi d \\ &= 3.141\ 592\ 7(4.35) \\ &= 13.665\ 928 \end{aligned}$$

Since the student measured the diameter and found it to be 4.3 cm to the nearest tenth of a centimetre, all he knows about the circumference is that it is between 13.351 769 cm and 13.665 928 cm. To overcome this discrepancy, we use the following convention.

> When calculating with measurements, the final answer should be written with the same number of significant digits used in the measurements.

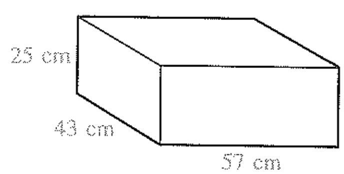
When the student calculated the circumference by multiplying his measured diameter 4.3 cm by π, he should have rounded the circumference to the same number of significant digits as his measurement. That is, his calculated result was 13.508 849, which he should have rounded to two significant digits. He should have written the answer as 14 cm.

There are two important things to realize about the above convention.

- The convention does not always give accurate results. For example, the student's circle could have had a diameter much closer to 4.25 cm than to 4.35 cm. The circumference would then be much closer to 13.351 769 cm than to 13.665 928 cm. To two significant digits, the circumference would be 13 cm and not 14 cm.
- Significant digits are used *only* when the numbers involved in a calculation are actual measurements. For example, we can visualize a circle having a diameter of *exactly* 4.3 cm. The circumference of this circle could be written as 13.508 849 cm, or even as 4.3π cm, without using the convention.

When you consider the significant digits in a number between 0 and 1, the leading zeros, which precede the non-zero digits, are not counted as significant digits. For example, each of the numbers 0.5, 0.006, and 0.000 007 has 1 significant digit.

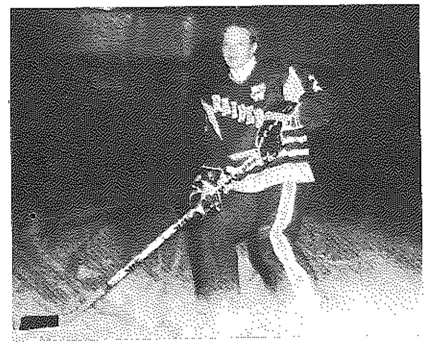
1. Use your calculator to find the circumferences of the circles with these diameters. Round your answers appropriately.
   a) 1.8 cm b) 0.09 m c) 0.6 km d) 15.7 cm e) 0.44 m
2. The dimensions of a rectangular solid are measured as 57 cm by 43 cm by 25 cm, to the nearest centimetre.



   a) Use these dimensions to calculate the volume of the solid. To how many significant digits should the volume be written?
   b) State the least possible value of each dimension.
   c) Calculate the least possible volume.
   d) State the greatest possible value of each dimension.
   e) Calculate the greatest possible volume.
   f) Express the volumes calculated in parts a), c), and e) to:
      i) 2 significant digits ii) 1 significant digit.
   g) To how many significant digits should the volume be expressed, to be representative of all possible values?

PROBLEM SOLVING

Solve a Simpler Related Problem



The city of Toronto organized a hockey league. However, the league organizer discovered that when the players were divided into teams of 6, teams of 7 or teams of 8, there was always one player left over. Finally another player joined the league and now the players could be divided into teams of 10, without left overs.
How many teams were in the league?

Understand the problem

- What do we know about the number of players in the league?
- Was the number of players in the league a multiple of 6, 7 or 8 before the last player joined?
- How is the number of players related to the number of teams?
- What are we asked to find?

Think of a strategy

- Try solving a simpler related problem, for example, what number leaves a remainder of 0 when divided by 6, by 7, and by 8?

Carry out the strategy

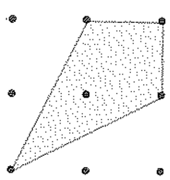
- Any number which leaves a remainder of 0 when divided by 6, 7 or 8 is a common multiple of 6, 7, and 8.
- To find the least common multiple of 6, 7, and 8, we write each number as a product of its prime factors.
  $6 = 2 \times 3$; $7 = 7$; $8 = 2^3$
- Then we form the product of all the prime factors to the highest power to which they occur. The least common multiple of 6, 7, and 8 is therefore $2^3 \times 3 \times 7$, or 168.
- All the common multiples of 6, 7, and 8 are of the form $168n$ where n is a integer.
- All numbers which leave a remainder of 1 when divided by 6, 7, or 8 are of the form $168n + 1$.
- The smallest positive integers which leave a remainder of 1 when divided by 6, 7, or 8 are: $168(1) + 1$, or 169; $168(2) + 1$, or 337; $168(3) + 1$, or 505.
- Of these, only 169 is a multiple of 10 when 1 is added. There were 170 players and hence 17 teams.

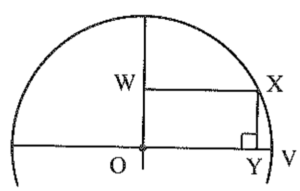
Look back

- List the smallest 6 numbers which leave a remainder of 1 when divided by 6, 7, or 8.
- Is 169 the only one of these which is a multiple of 10 when increased by 1?
- If there is another number which leaves a remainder of 1 when divided by 6, 7, or 8 and it is a multiple of 10 when increased by 1, why would it be rejected as an answer to the problem?

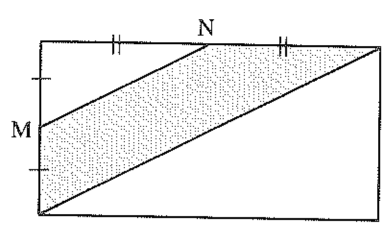
Solve each problem

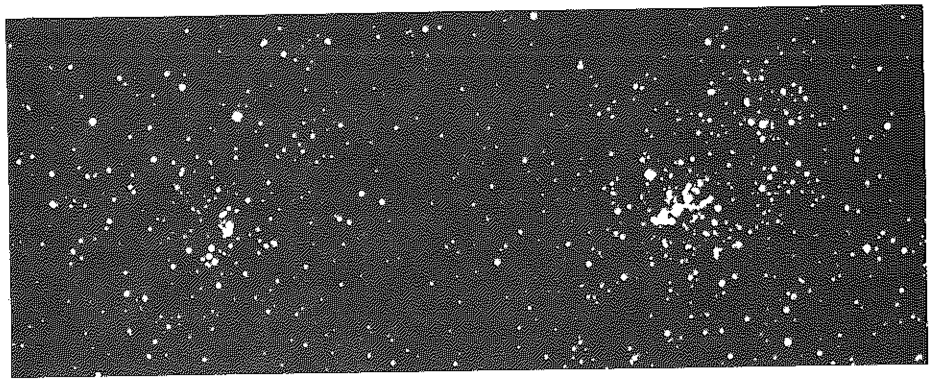
1. What time will it be:
   a) 24 000 h from now
   b) 23 999 992 h from now?
2. Let $N = 1 \times 2 \times 3 \times 4 \times 5 \times \ldots \times 19 \times 20$
   a) What is the largest power of 5 of which N is a multiple?
   b) What is the largest power of 2 of which N is a multiple?
   c) How many zeros come at the end of the numeral for N?
3. a) What is the smallest multiple of 300 which has all its prime factors to an even power?
   b) What is the smallest multiple of 300 which is the square of a positive integer?
   c) What is the smallest multiple of 300 which is the cube of a positive integer?
4. What is the area of the shaded kite drawn on 1 cm paper (below left)?





5. In rectangle WXYO, OY = 15 cm and YV = 2 cm. What is the length of WY if O is the centre of the circle (above right)?
6. M and N are the midpoints of the sides of a rectangle. What fraction of the rectangle is shaded?





5-5 SCIENTIFIC NOTATION

Scientists tell us that there are about 120 000 000 000 stars in our galaxy, the Milky Way. Only the first two digits in this number are significant, the zeros are place holders to show the position of the decimal point.

Large numbers like this are awkward to write and difficult to read. To express very large numbers (and very small numbers) more simply, we use *scientific notation*.

When a number is expressed in scientific notation, it is written as the product of:

- a number greater than or equal to 1 but less than 10, and
- a power of 10.

To express 120 000 000 000 in scientific notation, write the decimal after the first non-zero digit and drop the trailing zeros, to get 1.2.

Since the true position of the decimal is 11 places to the right, we multiply the number 1.2 by 10^{11}.

That is, $120\ 000\ 000\ 000 = 1.2 \times 10^{11}$

The mass of a hydrogen atom is 0.000 000 000 000 000 000 000 001 67 g. To express this very small number in scientific notation, write the decimal after the first non-zero digit and drop the preceding zeros, to get 1.67.

Since the true position of the decimal is 24 places to the left, we multiply the number 1.67 by 10^{-24}.

That is, $0.000\ 000\ 000\ 000\ 000\ 000\ 000\ 001\ 67 = 1.67 \times 10^{-24}$

Example 1. Simplify. $\dfrac{24\ 000\ 000\ 000 \times 0.000\ 02}{3200}$

Solution. Rewrite the expression using scientific notation.

$$\begin{aligned}\frac{24\ 000\ 000\ 000 \times 0.000\ 02}{3200} &= \frac{2.4 \times 10^{10} \times 2 \times 10^{-5}}{3.2 \times 10^{3}}\\ &= \frac{2.4 \times 2}{3.2} \times \frac{10^{10} \times 10^{-5}}{10^{3}}\\ &= 1.5 \times 10^{2}\\ &= 150\end{aligned}$$

Example 2. Write in scientific notation and estimate the answer to one significant digit.

$$\frac{389\ 527 \times 6\ 058\ 732}{4793.82}$$

Solution. Write each number in scientific notation to one significant digit.

$389\ 527 \doteq 4 \times 10^5$; $6\ 058\ 732 \doteq 6 \times 10^6$; $4793.82 \doteq 5 \times 10^3$

$$\frac{389\ 527 \times 6\ 058\ 732}{4793.82} \doteq \frac{4 \times 10^5 \times 6 \times 10^6}{5 \times 10^3}$$
$$\doteq 5 \times 10^8$$

Since the factors were rounded to one significant digit, the estimate should not exceed one significant digit.

Example 3. It has been estimated that if the average mass of an automobile were reduced from 1500 kg to 1000 kg, Canada would save about 30 000 000 L of oil each day.

a) Find how much oil Canada would save in one year.

b) Three litres of oil yield 2 L of gasoline. Assume that there are 9 000 000 cars on Canadian roads and gasoline costs 55¢/L. Find the annual saving per car.

Solution. a) Oil saved per year $= 365 \times 30\ 000\ 000$
$= 3.65 \times 10^2 \times 3.0 \times 10^7$
$= 10.95 \times 10^9$
$= 1.095 \times 10^{10}$

The annual saving of oil would be about 1.1×10^{10} L.

b) 1.095×10^{10} L of oil are saved every year.

Hence, $\frac{2}{3}(1.095 \times 10^{10})$ L of gasoline are saved every year.

There are 9 000 000 cars on the road.

Hence, the gasoline saved per year per car is

$$\frac{1}{9\ 000\ 000} \times \frac{2}{3}(1.095 \times 10^{10}) \text{ L.}$$

Gasoline costs \$0.55/L.

Hence, the annual saving per car is

$$\frac{1}{9\ 000\ 000} \times \frac{2}{3}(1.095 \times 10^{10}) \times 0.55 \text{ dollars}$$
$$= \frac{2 \times 1.095 \times 0.55 \times 10^{10}}{9 \times 3 \times 10^6}$$
$$\doteq 0.0446 \times 10^4$$
$$\doteq 446$$

The annual saving per car could be about \$450.

What would be the annual saving per car with the price of gasoline today?

EXERCISES 5-5

(A)

1. Write in scientific notation.
   a) 1000 b) 100 000 000 c) 100 d) 750
   e) 1100 f) 3 700 000 g) 0.0001 h) 0.000 000 1
   i) 0.000 001 j) 0.000 85 k) 0.000 092 l) 0.000 000 008 2
   m) 85 n) 0.038 o) 9900 p) 3 210 012

2. Write in scientific notation.
   a) Speed of light, 300 000 km/s
   b) World population in 1985, 4 843 000 000
   c) Mass of the Earth, 5 980 000 000 000 000 000 000 000 kg
   d) Time of fastest camera exposure, 0.000 000 1 s
   e) Mass of the ball in a ball-point pen, 0.004 g

3. What numbers complete this table?

	Physical Quantity	Decimal Notation	Scientific Notation
a)	Temperature of the sun's interior	1 300 000°C	
b)	Thickness of a plastic film	0.000 01 m	
c)	Mass of an electron		9.2×10^{-28} g
d)	Number of stars in our galaxy		1.2×10^{11}
e)	Estimated age of the Earth	4 500 000 000 years	
f)	Diameter of a hydrogen atom	0.000 000 011 3 cm	
g)	Land area of the Earth		1.5×10^{8} km²
h)	Ocean area of the Earth		3.6×10^{8} km²
i)	Mass of the Earth		5.9×10^{24} kg
j)	Cost of a Concorde aircraft	8 500 000 000 F	

(B)

4. Write in scientific notation.
   a) 32×10^4 b) 247×10^8 c) 49.2×10^7 d) 685×10^{10}
   e) 0.387×10^4 f) 0.087×10^3 g) 672×10^{-5} h) 43.7×10^{-6}
   i) 0.841×10^{-2} j) 0.49×10^{-7} k) 125×10^0 l) $1.85 \div 10^{-2}$

5. Find each value for n.
   a) $1265 = 1.265 \times 10^n$ b) $76.3 = 7.63 \times 10^n$
   c) $0.0041 = 4.1 \times 10^n$ d) $0.860 = 8.60 \times 10^n$
   e) $0.005 = 5 \times 10^n$ f) $0.000\,056\,3 = 5.63 \times 10^n$
   g) $1150 = 1.150 \times 10^n$ h) $4\,961\,000\,000 = 4.961 \times 10^n$
   i) $7\,430\,000 = 7.43 \times 10^n$ j) $0.000\,000\,583\,1 = 5.831 \times 10^n$

6. Express these distances and measurements in scientific notation.

From Galactic Distances to Atomic Measurements, all in metres

100 000 000 000 000 000 000 000 000	Distance to quasars
14 000 000 000 000 000 000 000	Distance to the nearest galaxy
760 000 000 000 000 000 000	Diameter of our galaxy
41 000 000 000 000 000	Distance to the nearest star
12 000 000 000 000	Diameter of the solar system
150 000 000 000	Distance to the sun
380 000 000	Distance to the moon
13 000 000	Diameter of the Earth
8 800	Height of Mount Everest
2	Height of a person
Size of an insect	0.005
Diameter of a grain of sand	0.000 1
Size of a bacterium	0.000 001
Diameter of an atom	0.000 000 000 1
Diameter of a nucleus	0.000 000 000 000 01
Diameter of a proton	0.000 000 000 000 000 1
Wavelength of cosmic rays	0.000 000 000 000 000 000 01

7. Write in scientific notation and estimate the answer.

a) $\dfrac{582\,965 \times 7\,123\,085}{5034.8}$ b) $\dfrac{9\,867\,341 \times 403\,928}{79\,386.3}$

c) $\dfrac{1\,937\,281 \times 8\,886\,432}{2916.5 \times 58\,034}$ d) $\dfrac{38\,621 \times 49\,728 \times 392.6}{79\,362 \times 193\,481}$

8. Simplify, and express the answer in scientific notation rounded to two significant digits.

a) $349\,000 \times 2650 \times 120\,000$ b) $8600 \times 1\,500\,000 \times 0.0003$

c) $\dfrac{480\,000 \times 62\,000\,000}{300\,000}$ d) $\dfrac{850\,000 \times 400\,000}{6\,200\,000}$

e) $\dfrac{0.000\,006 \times 54\,000}{0.000\,009}$ f) $\dfrac{2\,400\,000 \times 0.000\,000\,000\,8}{0.000\,000\,4 \times 12\,000}$

9. Write in scientific notation and estimate the answer.

a) $\dfrac{392\,876 \times 48\,731 \times 0.000\,186}{0.000\,007\,7 \times 3\,865\,097}$ b) $\dfrac{0.000\,000\,28 \times 78\,365\,294}{1873.6 \times 29\,586 \times 0.0038}$

c) $\dfrac{29\,307\,608 \times 30\,962 \times 567\,081}{0.000\,089 \times 5\,821\,939}$ d) $\dfrac{23\,501\,784 \times 0.000\,935}{0.000\,248\,6 \times 225.69}$

e) $\dfrac{0.000\,730\,8 \times 0.017\,41 \times 642}{52\,325 \times 0.002\,79 \times 48}$ f) $\dfrac{3127.98 \times 0.005\,294 \times 1.4372}{1001 \times 0.9842 \times 0.000\,55}$

10. Californium 252, one of the world's rarest metals, is used in treating cancer. If one-tenth of a microgram costs \$100, what is its cost per gram?

11. The measured daily deposit of the pollutant sulphur dioxide on Metropolitan Toronto is approximately 4.8×10^{-6} g/cm^2. If Metropolitan Toronto has an area of about 620 km^2, and the pollutant is distributed evenly, calculate the amount of sulphur dioxide that falls on the city:
    a) in 1 day b) in 1 year c) in your lifetime.

12. If it takes 1200 silkworm eggs to balance the mass of 1 g, what is the mass of one silkworm egg?

13. The volume of water in the oceans is estimated to be 1.35×10^{18} m^3. If the density of sea water is 1025 kg/m^3, what is the mass of the oceans?

14. A faucet is leaking at the rate of one drop of water per second. The volume of one drop is 0.1 cm^3.
    a) Calculate the volume of water lost in a year.
    b) Calculate how long it would take to fill a rectangular basin 30 cm by 20 cm by 20 cm.

15. In 1986, astronomers discovered a chain of galaxies, which stretches a billion light-years from one end to the other. It is the largest structure ever found in the universe. A light-year is the distance that light, with a speed of 300 000 km/s, travels in one year.
    a) Calculate the approximate number of kilometres in one light-year.
    b) Calculate the length, in kilometres, of the astronomers' discovery.

16. It has been estimated that the insect population of the world is at least 1 000 000 000 000 000 000. The scientist who made this estimate also reckoned that if the average mass of an insect is 2.5 mg, then the total mass of insects on the Earth is twelve times as great as the total mass of all human beings.
    a) Estimate the total mass of the Earth's insect population.
    b) Estimate the total mass of the Earth's human population.
    c) Use the population estimate in *Exercise* 2 and your answer to part b) to calculate the average mass of a person. Is this average reasonable?

17. A drop of oil with a volume of 1 mm^3 spreads out on the surface of water until it is a film one molecule thick. If the film has an area of 1 m^2, what is the thickness of an oil molecule?

18. The number 10^{100} is sometimes called a *googol*. To help you understand how large a number this is, think of the visible universe as a cube with edges of length 2×10^{25} m. Suppose this cube were filled with cubical grains of sand with edges of length 2×10^{-4} m. How many grains of sand would be needed?

19. Write in scientific notation.
    a) The square root of a googol
    b) The fourth root of a googol
    c) The tenth root of a googol

MATHEMATICS AROUND US

How Bacteria Grow and Multiply

At midnight, there were 1000 bacteria in a culture. The number of bacteria doubles every hour so n hours later there would be $1000(2^n)$. Does this statement make sense if n is negative?

If n were -3, it would be like asking, "How many bacteria were there -3 h after midnight?"; -3 h after midnight is 3 h *before* midnight. To find the number of bacteria 3 h before midnight, multiply 1000 by 2^{-3}.

$$\begin{aligned} 1000(2^{-3}) &= 1000\left(\frac{1}{8}\right) \\ &= 125 \end{aligned}$$

There were about 125 bacteria in the culture 3 h before midnight.

QUESTIONS

1. About how many bacteria were in the culture:
   a) 1 h before midnight b) 2 h before midnight?
2. At 00:30, there were about 1400 bacteria in the culture. About how many were there:
   a) 1 h earlier b) 2 h earlier c) 3 h earlier?
3. At 03:45, there were about 13 500 bacteria in the culture. About how many were there:
   a) 3 h earlier b) 4 h earlier c) 5 h earlier?

COMPUTER POWER

Scientific Notation

Most microcomputers can display nine digits in a number. When a computer displays a number in scientific notation, it is likely to use E to indicate that the number is in this form.

Most microcomputers display:

- numbers less than 1 billion in decimal form, for example, the command PRINT 999999999 yields the display 999999999.
- numbers 1 billion or greater in scientific notation, for example, the command PRINT 1000000000 yields the display 1E + 09; the command PRINT 1400000000 yields the display 1.4E + 09.

After each input, press [RETURN].

1. Write the display produced by each command.
   a) PRINT 690000000 b) PRINT 7200000000 c) PRINT 42800000000
   Use the computer to check your answers.
2. Determine the largest number of digits your computer will display.
3. Determine the smallest positive number your computer will display in decimal form.
4. Write the display produced by each command.
   a) PRINT 0.052 b) PRINT 0.0035 c) PRINT 0.000096
5. Write, in decimal form, the number expressed in each computer display.
   a) 1.7 E + 09 b) 9.63 E + 10 c) 7.19 E + 12
   d) −6.2 E + 11 e) 9.3 E − 03 f) 3.05 E − 08
6. Input each command on your computer. Record each answer.
   a) PRINT 1E+09*1E+10 b) PRINT 1E+09*1E−08
   c) PRINT 1E+10*1E−13 d) PRINT 1E+22/1E+10
   e) PRINT 1E+08/1E−03 f) PRINT 1E+12/1E−03
7. Study your answers in *Exercise 6*. Can you write a simple rule for:
   a) multiplying powers of 10 b) dividing powers of 10?
8. Use your rules in *Exercise 7* to predict these products and quotients.
   a) 1E+11*1E+15 b) 1E+18*1E−08
   c) 1E+29/1E+16 d) 1E+31/1E+14
   Use the computer to check your answers.
9. What happens if:
   a) you enter 2E12 instead of 2E+12
   b) you enter 1E−8 instead of 1E−08
   c) you enter E−9 instead of 1E−09

5-6 SQUARE ROOTS

When a skydiver leaves an airplane, the distance fallen during the first few seconds of free fall is related to the time that has elapsed by the formula $t^2 \doteq \frac{1}{5}d$. d is the distance in metres and t is the time in seconds.

How long does it take the skydiver to fall a distance of 80 m?
To find the value of t, substitute $d = 80$ into the equation.

$$t^2 \doteq \frac{1}{5}(80)$$
$$= 16$$

t^2 means $t \times t$. To solve the equation, we need to find the value of t that multiplied by itself gives 16.

We know that $4 \times 4 = 16$ and $(-4) \times (-4) = 16$.
Hence t must equal 4 or -4.

But a negative value for t has no meaning in this context.

Hence, the solution is $t = 4$.

The skydiver takes about 4 s to fall 80 m.

When a number (like 16) can be written as the product of two factors that are the same, each factor is a *square root* of that number.

That is, since $4 \times 4 = 16$ and $(-4) \times (-4) = 16$, then 4 and -4 are the square roots of 16.

Similarly, since $7 \times 7 = 49$ and $(-7) \times (-7) = 49$, then 7 and -7 are the square roots of 49.

Further, since $0.2 \times 0.2 = 0.04$ and $(-0.2) \times (-0.2) = 0.04$, then 0.2 and -0.2 are the square roots of 0.04.

Positive numbers always have two square roots — one positive, the other negative. The symbol, $\sqrt{}$, is the *radical sign* and it always denotes the positive square root.

Thus, $\sqrt{49} = 7$ and $\sqrt{0.04} = 0.2$

Example 1. Evaluate.

a) $\sqrt{121}$ b) $-\sqrt{1.21}$ c) $-\sqrt{12\,100}$ d) $\sqrt{0.0121}$

Solution.

a) Since $11 \times 11 = 121$, then $\sqrt{121} = 11$

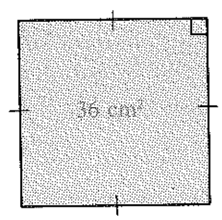
b) Since $1.1 \times 1.1 = 1.21$, then $-\sqrt{1.21} = -1.1$

c) Since $110 \times 110 = 12\,100$, then $-\sqrt{12\,100} = -110$

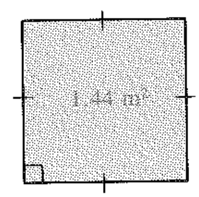
d) Since $0.11 \times 0.11 = 0.0121$, then $\sqrt{0.0121} = 0.11$

Example 2. Find the length of the side of each square.

a)



b)



Solution. The area A of a square is equal to the square of the length of the side s.

That is, $A = s^2$

a) If $A = 36$, then $s = \sqrt{36}$
$= 6$

The side of the square is 6 cm.

b) If $A = 1.44$, then $s = \sqrt{1.44}$

We know that $\sqrt{144} = 12$, so $\sqrt{1.44} = 1.2$.

The side of the square is 1.2 m.

Radical signs are usually treated like brackets. Operations under radical signs are performed first.

Example 3. Evaluate.

a) $-3\sqrt{6.25}$ b) $\sqrt{9 + 16}$ c) $5\sqrt{4} - 3(\sqrt{121} - \sqrt{81})$

Solution.

a) $-3\sqrt{6.25} = -3(2.5)$
$= -7.5$

b) $\sqrt{9 + 16} = \sqrt{25}$
$= 5$

c) $5\sqrt{4} - 3(\sqrt{121} - \sqrt{81}) = 5(2) - 3(11 - 9)$
$= 10 - 3(2)$
$= 10 - 6$
$= 4$

Example 4. Evaluate each expression for $x = -2$ and $y = 3$.

a) $\sqrt{-x + y + 4}$ b) $-\sqrt{6x + 5y - xy}$

Solution. Substitute the given values for x and y.

a) $$\begin{aligned}\sqrt{-x + y + 4} &= \sqrt{-(-2) + 3 + 4}\\ &= \sqrt{2 + 3 + 4}\\ &= \sqrt{9}\\ &= 3\end{aligned}$$

b) $$\begin{aligned}-\sqrt{6x + 5y - xy} &= -\sqrt{6(-2) + 5(3) - (-2)(3)}\\ &= -\sqrt{-12 + 15 + 6}\\ &= -\sqrt{9}\\ &= -3\end{aligned}$$

EXERCISES 5-6

Ⓐ

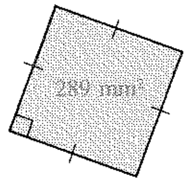
1. Find the square roots of each number.
   a) 81 b) 10 000 c) 900 d) 0.16 e) 14 400
   f) 40 000 g) 0.64 h) 0.0001 i) 4900 j) 0.25
2. Evaluate.
   a) $\sqrt{49}$ b) $-\sqrt{0.04}$ c) $\sqrt{1600}$ d) $\sqrt{169}$ e) $-\sqrt{3600}$
   f) $\sqrt{1.44}$ g) $-\sqrt{225}$ h) $\sqrt{10^{12}}$ i) $\sqrt{625}$ j) $-\sqrt{2^4}$
3. Find the side length of each square with the given area.
   a) $16\ m^2$ b) $10\ 000\ mm^2$ c) $6.25\ cm^2$
   d) $2^6\ m^2$ e) $10^4\ m^2$ f) $4900\ m^2$

Ⓑ

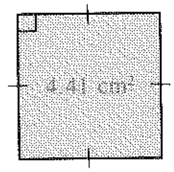
4. Simplify.
   a) $\sqrt{64 + 36}$ b) $\sqrt{16} + \sqrt{9}$ c) $2\sqrt{16} - 3\sqrt{4}$
   d) $3\sqrt{36} + 2\sqrt{25}$ e) $5\sqrt{100 - 36}$ f) $\sqrt{1 + 3 + 5 + 7 + 9}$
   g) $2\sqrt{81} - 7\sqrt{49}$ h) $4\sqrt{289 - 225}$ i) $2\sqrt{\sqrt{81}}$
5. Evaluate each expression for $a = 5$ and $b = -3$.
   a) $\sqrt{20a}$ b) $\sqrt{9a^2}$ c) $\sqrt{\frac{125}{a}}$
   d) $\sqrt{2a - 13b}$ e) $\sqrt{-12b}$ f) $-\sqrt{3a - 3b + 1}$
   g) $\sqrt{a^2 + 3b}$ h) $\sqrt{7a - 8b + 5}$ i) $-4\sqrt{11a - 3b}$
   j) $-3\sqrt{2a^2 - 4b^2 + 2}$ k) $0.5\sqrt{a^2 - 2ab + b^2}$ l) $2\sqrt{2a^2 - 3b^2 + 2}$
6. Evaluate each expression for $x = 3$, $y = -4$, and $z = -7$.
   a) $-\sqrt{12x}$ b) $\sqrt{z^2 + 6y}$ c) $4\sqrt{x^2 + y^2}$
   d) $\sqrt{6x - z}$ e) $5\sqrt{15x - y}$ f) $-\sqrt{7x - 4y - 1}$
   g) $2\sqrt{x - 2y - 2z}$ h) $-\sqrt{2z^2 + 5y + x}$ i) $6\sqrt{3y^2 + x^0}$
   j) $\sqrt{-(-2x + 4y + 2z)}$ k) $\sqrt{x^2 + 2x + z^2}$ l) $-\sqrt{3x^2 + y + 2z^2}$

7. From the area of each square, calculate:
   i) the length of a side
   ii) the perimeter.

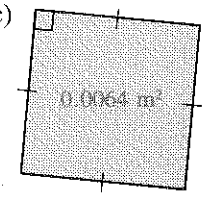
a)



b)



c)

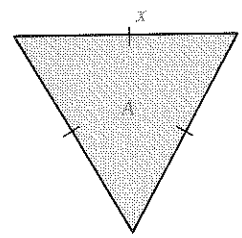


8. The distance d metres that an object falls from rest in t seconds is given by the formula $d \doteq 5t^2$. A pebble is dropped from a cliff 320 m high. Find how long the pebble takes to reach the ground.

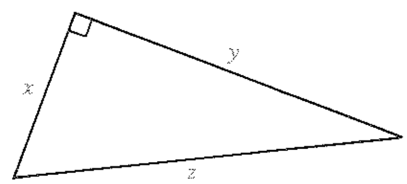
9. For an equilateral triangle of side length x, the area A is given by $A \doteq 0.43x^2$. For each equilateral triangle with the given area, calculate:
   i) the length of a side
   ii) the perimeter.

   a) 10.825 m^2 b) 27.712 m^2
   c) 0.350 73 m^2 d) 389.7 cm^2
   e) 0.004 33 km^2 f) 97.425 km^2

   Give the answers to 2 decimal places.



10. In a right triangle with side lengths as shown, the formula $z^2 = x^2 + y^2$ applies. Calculate z for each value of x and y.
    a) 6 mm, 8 mm
    b) 8 mm, 15 mm
    c) 0.5 m, 1.2 m
    d) 24 mm, 7 mm
    e) 0.03 km, 0.04 km
    f) 2.8 km, 9.6 km



11. In winter, you should have noticed that the stronger the wind the colder it feels. Winter weather forecasts often give "wind-chill" temperatures.

    For wind speeds of 10 km/h and higher, the following formula gives an approximate value for the wind-chill temperature.
    $w = 33 - (0.23\sqrt{v} + 0.45 - 0.01v)(33 - T)$
    w is the wind-chill temperature in degrees Celsius.
    v is the wind speed in kilometres per hour.
    T is the still-air temperature in degrees Celsius.
    For a still-air temperature of -18°C, calculate the wind-chill temperatures, to the nearest degree, at these wind speeds.
    a) 100 km/h   b) 80 km/h
    c) 30 km/h   d) 15 km/h

5-7 ESTIMATING AND CALCULATING SQUARE ROOTS

Renata was standing on the observation level of the CN Tower. It was a clear day and she could see to the horizon. She wondered how far away the horizon was.

The formula $d \doteq 3.6\sqrt{h}$ relates the distance d kilometres to the horizon from an observer who is h metres above the ground.

The observation level of the CN Tower is 457 m above the ground.

To find the distance to the horizon, substitute $h = 457$ into the formula $d \doteq 3.6\sqrt{h}$.

Then $d \doteq 3.6\sqrt{457}$

The number 457 has a square root which cannot be written as a terminating decimal. However, an approximation to its value can be found by estimating.

We know that $\sqrt{400} = 20$ so $\sqrt{457}$ will be a little more than 20.

Estimate. $\sqrt{457} = 21$

Check. $21^2 = 441$ so 21 is too small

Try 22. $22^2 = 484$ so 22 is too large

Try 21.5. $21.5^2 = 462.25$ so 21.5 is too large but not by much

Try 21.4. $21.4^2 = 457.96$ so 21.4 is very close

$\sqrt{457}$ is 21.4 to 1 decimal place.

Hence, the distance to the horizon, $d \doteq 3.6(21.4)$
$= 77.04$

Renata could see about 77 km to the horizon from the observation level of the CN Tower.

Since a calculator has a $\boxed{\sqrt{}}$ key, the approximate square root of a number can be found easily.

To evaluate $\sqrt{457}$, key in: $\boxed{4}$ $\boxed{5}$ $\boxed{7}$ $\boxed{\sqrt{}}$ to display 21.377558

Example 1. Find, by estimating and checking, the square roots of 8356 to the nearest whole number.

Solution. Start with numbers whose squares are familiar.

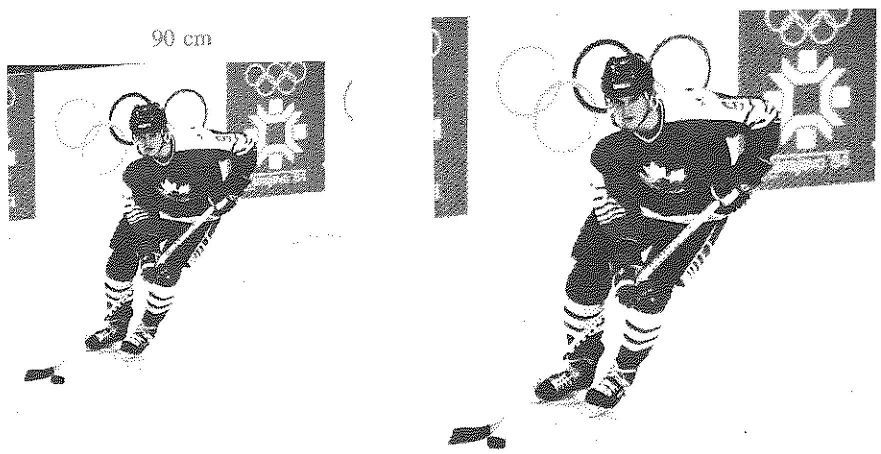
$100^2 = 10\ 000$ too high
$90^2 = 8100$ too low but close
$92^2 = 8464$ too high
$91.5^2 = 8372.25$ too high
$91.4^2 = 8353.96$ too low

By inspection, the square root of 8356 is closer to 91.4 than it is to 91.5.

Hence, the square roots of 8356 are 91 and -91 to the nearest whole number.

Consider the problem that was posed at the beginning of the chapter.

Example 2. Find, to the nearest centimetre, the length of each side of the square poster on the right, which is twice the area of the poster on the left.



Solution. Area of the smaller poster $= 90^2$
$= 8100$

Area of the larger poster $= 2(8100)$
$= 16\ 200$

The length of each side of a square is equal to the square root of its area.

Side of larger poster $= \sqrt{16\ 200}$
$\doteq 127.28$

Each side of the larger poster is approximately 127 cm long.

Although the area has doubled, the length of the side did not double.

EXERCISES 5-7

(A)

1. State which square roots are between 8 and 9.
$\sqrt{67}$, $\sqrt{91}$, $\sqrt{78}$, $\sqrt{62}$, $\sqrt{80}$

2. State which of the square roots listed below are between:
a) 3 and 4 b) 7 and 8 c) 11 and 12
d) 10 and 11 e) 13 and 14 f) 18 and 19.

$\sqrt{11}$, $\sqrt{52}$, $\sqrt{61}$, $\sqrt{14}$, $\sqrt{330}$, $\sqrt{360}$, $\sqrt{320}$, $\sqrt{257}$,
$\sqrt{190}$, $\sqrt{140}$, $\sqrt{171}$, $\sqrt{118}$, $\sqrt{110}$, $\sqrt{130}$, $\sqrt{80}$, $\sqrt{35}$

3. State which of the three estimates is closest to the square root.
a) $\sqrt{39}$; 6.2, 6.4, 6.6 b) $\sqrt{119}$; 10.3, 10.6, 10.9 c) $\sqrt{172}$; 13.1, 13.3, 13.5

(B)

4. Estimate each square root to one decimal place.
a) $\sqrt{18}$ b) $\sqrt{7.7}$ c) $\sqrt{111}$ d) $\sqrt{1473}$

5. Estimate each square root to one decimal place.
a) $\sqrt{29}$ b) $\sqrt{2.9}$ c) $\sqrt{14}$ d) $\sqrt{6.5}$ e) $\sqrt{43.5}$

6. Calculate the length, to one decimal place, of a side of each square with the given area.
a) 43.67 cm^2 b) 2.81 m^2 c) 9.48 cm^2 d) 37.37 m^2

7. Estimate each square root to the nearest unit.
a) $\sqrt{290}$ b) $\sqrt{1437}$ c) $\sqrt{175}$ d) $\sqrt{640}$ e) $\sqrt{8333}$

8. Use a calculator to find each square root to two decimal places.
a) $\sqrt{34.72}$ b) $\sqrt{21.38}$ c) $\sqrt{150.46}$ d) $\sqrt{0.62}$ e) $\sqrt{0.05}$

9. What happens when the $\boxed{\sqrt{\ }}$ key is used in an attempt to evaluate $\sqrt{-3}$? Explain why.

10. a) If you knew the area of a square, how could you find its perimeter?
b) Write a formula relating the perimeter P and the area A of a square.
c) Find the perimeters of the squares with these areas.
i) 25 cm^2 ii) 64 cm^2 iii) 78 m^2 iv) 3.8 m^2

11. a) If you knew the diameter of a circle, how could you find its area?
b) Write a formula for the area A of a circle in terms of its diameter d.
c) Find the areas of circles with these diameters. Give the answers to the nearest square unit.
i) 10 cm ii) 20 cm iii) 31.8 cm iv) 4 m
d) If you knew the area of a circle, how could you find its diameter?
e) Write a formula for the diameter of a circle in terms of its area.
f) Find the diameters of the circles with these areas. Give the answers to 1 decimal place.
i) 22 cm^2 ii) 33 cm^2 iii) 40 cm^2 iv) 5 m^2

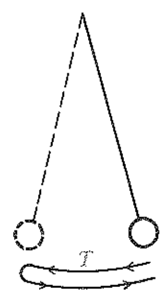
12. The distance d kilometres of the horizon from an observer at a height h metres is given by the formula $d \doteq 3.6\sqrt{h}$. To the nearest kilometre how far is the horizon from:
    a) the 266 m observation level of the Skylon tower in Niagara Falls
    b) eye level (at a height of 1.5 m) when an observer is standing on the ground?

Ⓒ

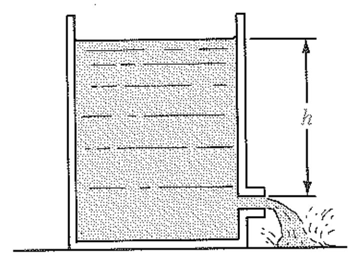
13. An object falls a distance d metres when falling from rest for t seconds. The relationship between d and t is given by the formula $d \doteq 4.9t^2$. If it takes 3 s to fall a certain distance, how long, to the nearest tenth of a second, will it take to fall:
    a) twice as far
    b) five times as far
    c) half as far
    d) ten times as far?

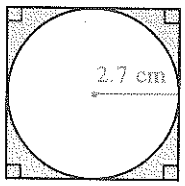
14. The length d of the diagonal of a rectangle of length l and width w is given by the formula $d = \sqrt{l^2 + w^2}$.
    a) Calculate the lengths of the diagonals of rectangles with these dimensions.
       i) 8 cm by 3 cm
       ii) 2.3 cm by 1.2 cm
       iii) 10 m by 7 m
       iv) 2 m by 9 m
    b) A rectangle is 9.2 cm long and its diagonal measures 11.5 cm. What is its width?

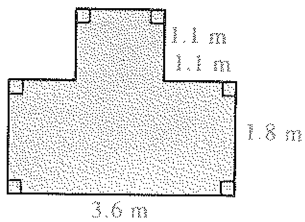
15. The period of a pendulum is the time taken for one complete swing to and fro. The period T seconds is given by the formula $T \doteq 2\pi\sqrt{\frac{l}{9.8}}$, where l is the length of the pendulum in metres.
    a) Find the period, to the nearest tenth of a second, of a pendulum whose length is:
       i) 2.45 m
       ii) 0.5 m.
    b) How long, to the nearest centimetre, is a pendulum whose period is 1 s?

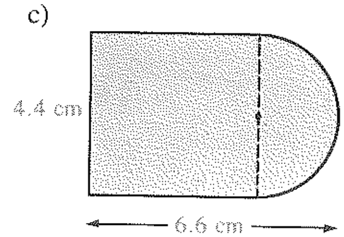


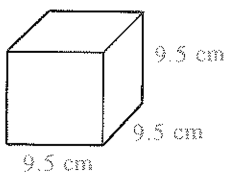
16. The velocity v metres per second with which liquid discharges from a small hole in a container is given by the formula $v \doteq \sqrt{19.6\,h}$, where h is the height in metres of the liquid above the hole.
    a) Find the velocity, to the nearest tenth of a unit, of discharge for each height.
       i) 1.0 m
       ii) 0.5 m
       iii) 10 cm
    b) What height, to the nearest centimetre, of liquid gives a discharge velocity of 2 m/s?

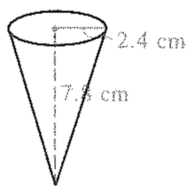


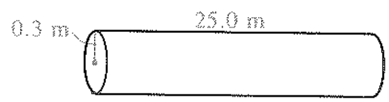
1. Evaluate.
   a) $3^2 + 2^3$ b) $3^2 \times 2^3$ c) $(1.6)^2$ d) $3^2 + 4$
   e) $(-2)^3 + (-3)^2$ f) $(2 + 3)^3$ g) $(-4)^2 + (-2)^4$ h) $\left(-\frac{3}{7}\right)^3$
2. If $x = -2$ and $y = 3$, evaluate each expression.
   a) $-5x^2$ b) $(-5x)^2$ c) $y^2 - x^2$ d) $(x + y)^3$
   e) $4x^2 + 3y^2$ f) $\frac{x^2 - y^2}{x - y}$ g) $2(x + y)^2$ h) $\frac{y^2 - x^3}{y - x}$
3. Find the area of the shaded region of each figure. Give the answers to 1 decimal place.
   a)


   b)


   c)


4. Find the volume of each solid. Give the answers to 1 decimal place.
   a)


   b)


   c)


5. Simplify.
   a) $x^4 \times x^5$ b) $x^{36} \div x^{12}$ c) $x^{12} \times x^6$ d) $(x)^7$
   e) $3x^2 \times 5x^4$ f) $2^9 \div 2^4$ g) $9m^4 \times 3m^2$ h) $(-3)^{12} \div (-3)^4$
   i) $(5x)^2$ j) $5(x^2)^3$ k) $(3x^2)^3$ l) $-42y^{12} \div 6y^8$
6. Simplify.
   a) $\frac{18x^4 \times 5x^2}{15x^3}$ b) $\frac{120x^5}{-15x} \times \frac{15x^4}{5x^2}$
7. Evaluate each expression for $x = 4$.
   a) $x^2 \times x^4$ b) $(2x^2)^2$ c) $(2x^2)(3x^3)$
8. Evaluate.
   a) 5^{-3} b) $\left(\frac{1}{2}\right)^{-1}$ c) $2^{-3} - 4^{-1}$ d) $7^0 + 2^{-2}$
   e) $3^2 - 3^{-2}$ f) $\left(\frac{2}{3}\right)^{-1} + \left(\frac{2}{3}\right)^0$ g) $\left(\frac{1}{2}\right)^{-2} + 2^{-1}$ h) $\left(\frac{1}{2}\right)^{-2} \div 2^{-1}$

9. Simplify.
   a) $w^8 \div w^{-4}$ b) $w^{-9} \div w^{-12}$ c) $15x^4 \div (-3x^{-4})$
   d) $-24y^4 \div 3y^{-2}$ e) $\frac{10}{y^4} \div y^2$ f) $16z^{-2} \div (2z)^2$

10. Evaluate each expression for $a = -2$, $b = 2$, and $c = -1$.
    a) $a^{-1} + b^{-1}$ b) $(a + b + c)^{-1}$ c) a^b d) $a^{-1} + b^{-1} + c^2$

11. Write in scientific notation.
    a) 10 000 b) 740 000 c) 0.000 01 d) 0.057

12. Express in scientific notation, then simplify.
    a) 49 000 000 × 730 000
    b) 26 500 000 × 7900 × 0.0046
    c) $\frac{320\ 000 \times 64\ 000\ 000}{12\ 800\ 000}$

13. Simplify.
    a) $\sqrt{36}$ b) $-\sqrt{0.25}$ c) $\sqrt{14\ 400}$ d) $\sqrt{0.0081}$
    e) $\sqrt{0.36}$ f) $-\sqrt{640\ 000}$ g) $\sqrt{0.0121}$ h) $-\sqrt{0.49}$

14. Simplify.
    a) $-3\sqrt{25} - 5\sqrt{9}$ b) $\sqrt{225} - \sqrt{49}$ c) $4\sqrt{169} - \sqrt{25}$

15. Find the side length of each square with the given area.
    a) 64 mm^2 b) 0.81 m^2 c) 49 cm^2 d) 2.25 cm^2

16. Calculate each square root to two decimal places.
    a) $\sqrt{28}$ b) $\sqrt{17.4}$ c) $\sqrt{250}$ d) $\sqrt{0.44}$

17. A square, with a side length of 10 cm, is to be reduced in area by one-half. Find the side length, to the nearest centimetre, of the reduced square.

18. Evaluate each expression for $m = 3$ and $n = -2$.
    a) $\sqrt{9m^2}$ b) $\sqrt{-8n}$ c) $-\sqrt{m + n}$
    d) $\sqrt{10m + 2n - 1}$ e) $-\sqrt{m^2 - 3n + 1}$ f) $\sqrt{m^3 + 2mn + 1}$

19. The approximate velocity v metres per second of an orbiting satellite is given by the formula $v = \sqrt{9.8r}$, where r is the distance in metres of the satellite from the centre of the Earth. Find the velocity of the satellite if:
    a) $r = 2 \times 10^7$ m b) $r = 5 \times 10^8$ m.

20. When a ball is dropped from a height of 2 m, the height h metres to which it bounces is given by $h = 2(0.8)^n$, where n is the number of bounces. To what height does the ball bounce after:
    a) the first bounce b) the third bounce?

21. The width w of a rectangle with length l and diagonal length d is given by the formula $w = \sqrt{d^2 - l^2}$. Find the width of a rectangle with length 12.2 cm and a diagonal of length 15.8 cm.