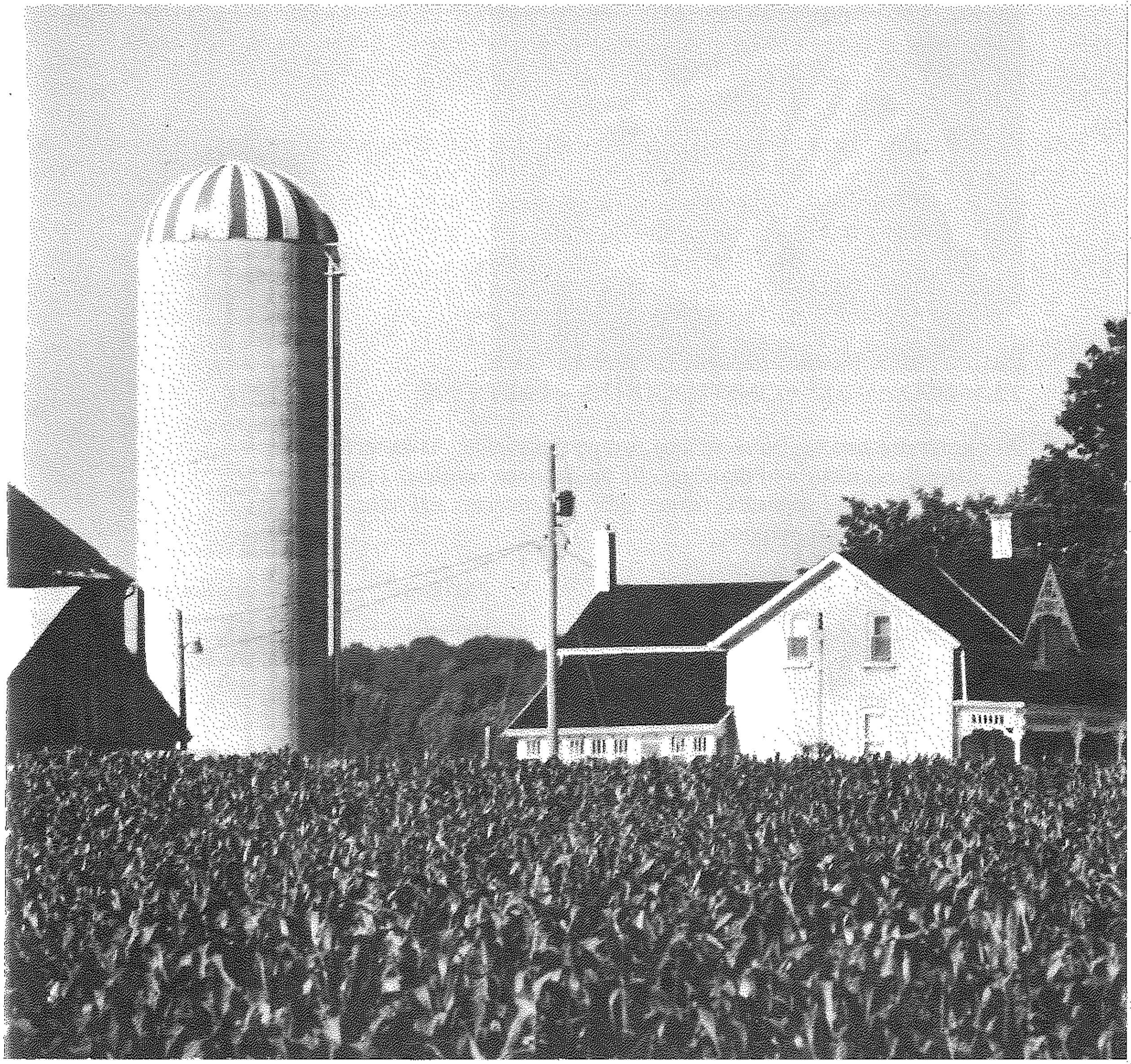
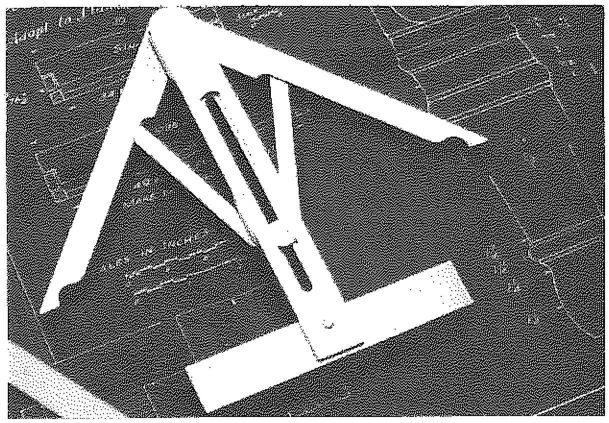
12 Geometric Constructions and 3-Dimensional Geometry



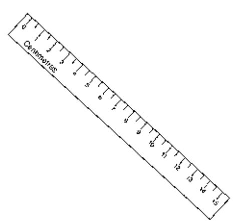
Why are silos built with circular bases rather than square ones? (See Section 12-6.)



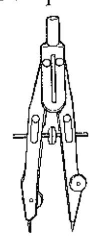
12-1 GEOMETRIC CONSTRUCTIONS – PART ONE

Any great structure begins as an idea in a person's mind. As an architect translates these ideas into drawings, he or she must construct certain lengths, angles, and geometric figures. Many instruments and techniques are available for this purpose.

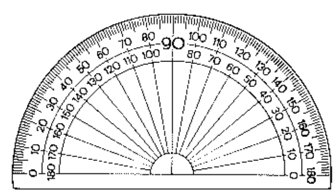
Ruler



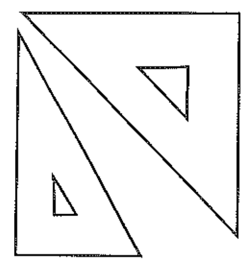
Compasses



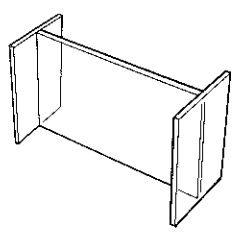
Protractor



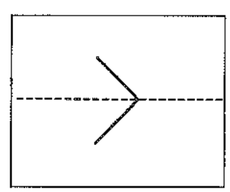
Set squares



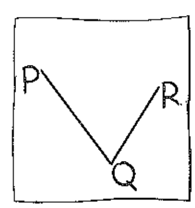
Transparent mirror



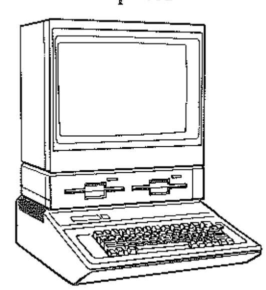
Paper folding



Tracing paper



Computer

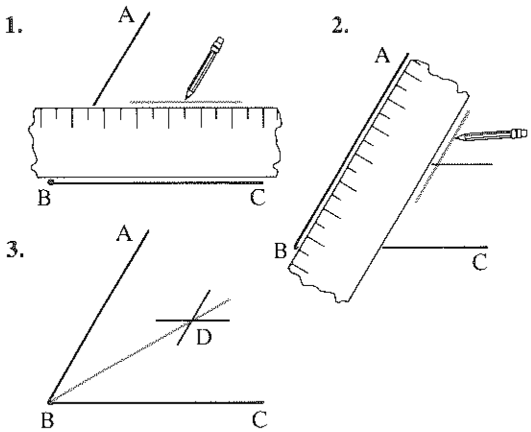


Today, most architectural drawings are prepared by a computer using a process called *computer assisted design*. The person who generates the drawing by computer is using a tool that is much more advanced than ruler and compasses. However, the mathematical principles of these constructions are unchanged.

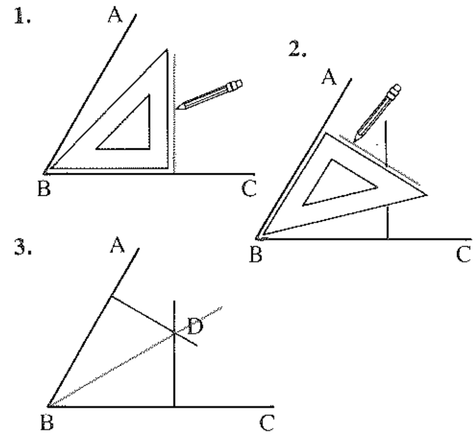
Constructing the bisector of an angle

We can use geometrical instruments in a variety of ways to construct the bisector of an angle.

Using two sides of a ruler

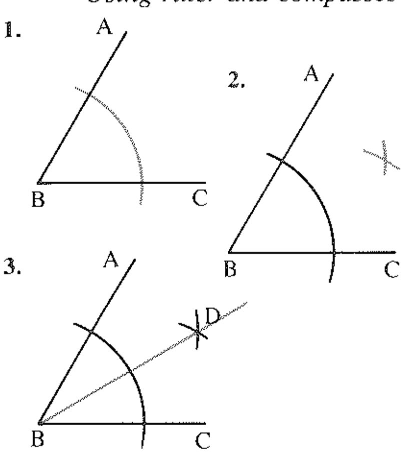


Using a set square

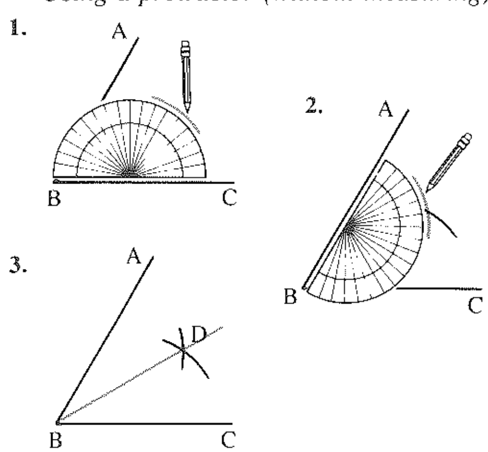


BD bisects ∠ABC.

Using ruler and compasses



Using a protractor (without measuring)



BD bisects ∠ABC.

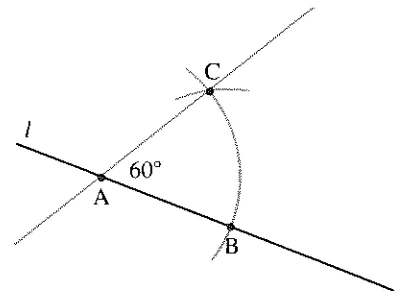
In each construction, we do the same thing on each arm of the angle. Hence, the constructions work because there is no reason for the line BD to be closer to one arm than to the other.

INVESTIGATE

Can you construct the bisector of an angle in other ways? How many different ways can you find?

Example. Using ruler and compasses, construct a 60° angle.

Solution. Draw a line l and mark point A on l.
Place the compasses point on A and draw part of a circle to cross the line at B.
Without changing the setting of the compasses, place the compasses point on B and draw an arc intersecting the part circle at C.
Join AC. $\angle CAB = 60°$



What other 60° angles could be formed by drawing lines on the diagram in this *Example*? Why does the method in this *Example* yield a 60° angle?

EXERCISES 12-1

Ⓐ

1. a) Draw an acute angle and bisect it.
   b) Draw an obtuse angle and bisect it.

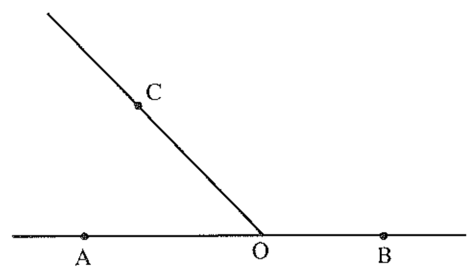
2. Construct a 60° angle and bisect it. Check by measuring.

Ⓑ

3. Draw any angle and divide it into four equal parts.

4. Construct each angle.
   a) 60° b) 30° c) 15° d) 120° e) 150°

5. Copy this diagram.
   a) Construct OD, the bisector of $\angle COB$.
   b) Construct OE, the bisector of $\angle AOC$.
   c) Measure $\angle DOE$.
   d) If the construction were repeated using a different position of line segment OC, would the answer to part c) be the same? Explain.



6. a) Construct $\triangle ABC$ in which AB = 7.5 cm, BC = 9.0 cm, and CA = 6.0 cm.
   b) Construct the bisectors of $\angle B$ and $\angle C$.
   c) Extend the bisectors in part b) to meet at D. Measure $\angle BDC$.

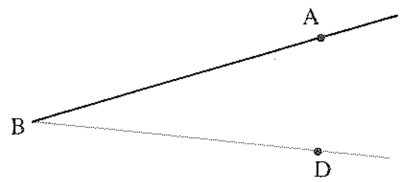
7. a) Construct $\triangle RST$ in which $\angle S = 60°$, ST = 6.0 cm, and RS = 10.0 cm.
   b) Measure $\angle R$, $\angle T$, and RT.
   c) Construct the bisector of $\angle RTS$.
   d) Extend the bisector in part b) to meet RS at U. Measure TU and $\angle TUR$.

8. Triangle ABC has vertices A(3, 8), B(−3, 0), and C(9, 0).
   a) Draw △ABC on a grid.
   b) Construct the bisector of ∠B and extend it to meet AC at D.
   c) Measure ∠ADB and ∠CDB.

9. a) Try to construct a triangle with these side lengths.
      i) 3 cm, 4 cm, 7 cm   ii) 5 cm, 5 cm, 12 cm
   b) What conclusions can you make?

10. a) Construct a regular hexagon with sides of length 4.0 cm.
    b) Measure a diagonal of the hexagon.

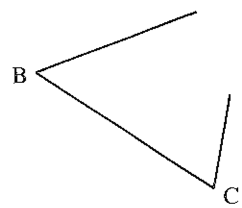
11. Copy this diagram. AB is one arm of ∠ABC, and BD is the bisector of ∠ABC. Construct the other arm of ∠ABC.



12. Draw an acute angle. Construct an angle with double the measure of the acute angle you drew.

13. Using only a ruler, construct three angles with a sum of 180°.

14. Copy this diagram. Using only a ruler, construct a third angle equal to the sum of ∠B and ∠C.

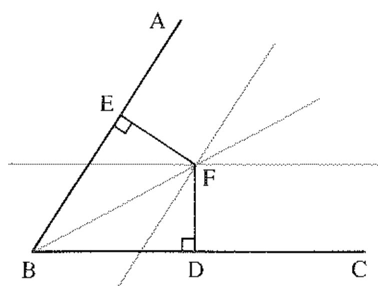


15. a) Construct an isosceles △ABC such that AC = BC.
    b) Extend AC to D.
    c) Construct CE, the bisector of ∠BCD.
    d) What can be said about CE and AB? Explain your answer.

16. Explain the construction for the bisector of an angle which uses a protractor without measuring.

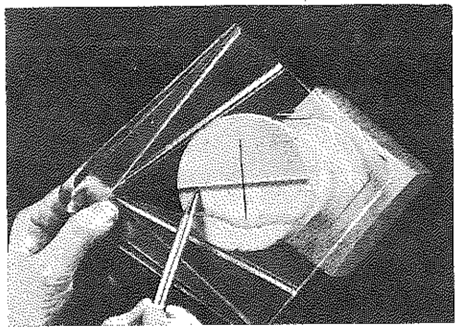
Ⓒ

17. The diagram shows the construction for the bisector of ∠ABC using two sides of a ruler.
    a) Explain why:
       i) FD = FE
       ii) BD = BE
       iii) △FDB ≅ △FEB.
    b) How does part a) explain why FB is the bisector of ∠ABC?



18. Explain the construction for the bisector of an angle which uses a set square.

19. The photograph on page 452 shows an *angle bisector* used by carpenters to bisect angles, and to lay out picture frames.
    a) How is the angle bisector used?
    b) Explain why it works.



12-2 GEOMETRIC CONSTRUCTIONS – PART TWO

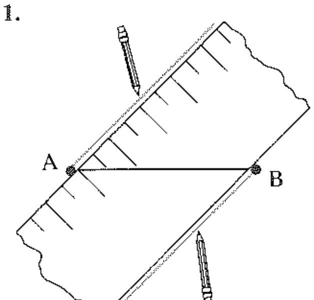
In the preceding section we illustrated four of many ways to construct the bisector of an angle. In this section we present methods of performing other constructions. Each construction can be done in many different ways.

Constructing the perpendicular bisector of a line segment

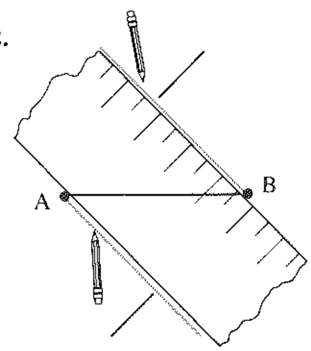
Infinitely many lines can be drawn perpendicular to a given line segment AB, but only one of them passes through the midpoint of AB. This line is called the *perpendicular bisector* of AB. Here are two methods of constructing the perpendicular bisector of AB.

Using two sides of a ruler

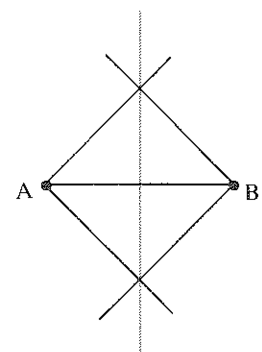
1.



2.

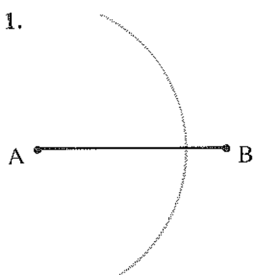


3.

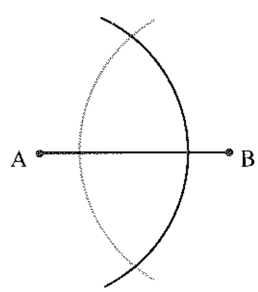


Using ruler and compasses

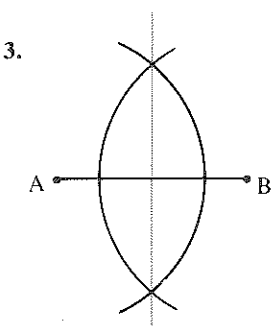
1.



2.



3.



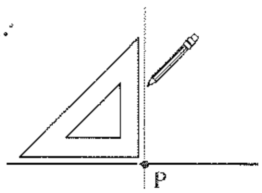
INVESTIGATE

What other ways can you find to construct the perpendicular bisector of a line segment?

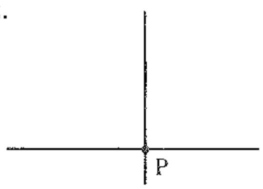
Constructing the perpendicular at a point on a line

Using a set square

1.



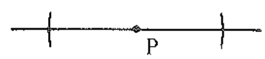
2.



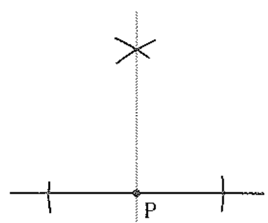
Using ruler and compasses

1.

2.



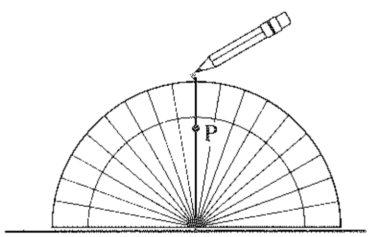
3.



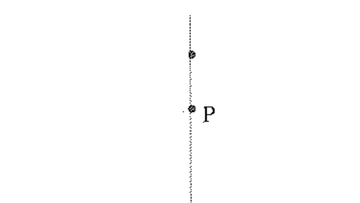
Constructing the perpendicular from a point to a line

Using a protractor

1.

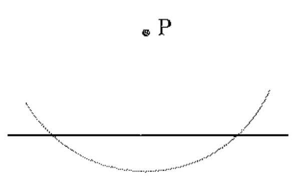


2.

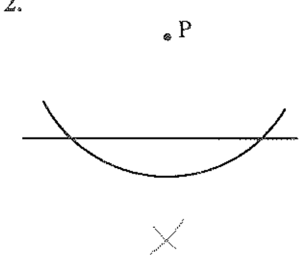


Using ruler and compasses

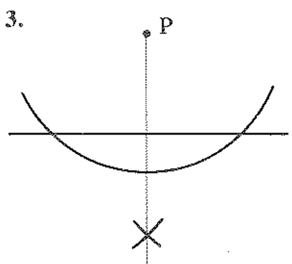
1.



2.



3.

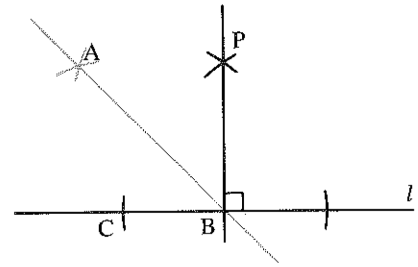


INVESTIGATE

What other ways can you find to construct perpendiculars at a point on a line or from a point to a line?

Example. Using a straightedge and compasses, construct a 45° angle.

Solution. Draw a line l.
Construct a perpendicular PB to l at any point B on l.
Then, $\angle PBC = 90°$.
Bisect $\angle PBC$. Line BA is the bisector of $\angle PBC$.
Hence, $\angle PBA = \angle ABC = 45°$



EXERCISES 12-2

(A)

1. Draw a line l. Choose any point A on l. Construct the perpendicular to l at A.
2. Draw a line l. Choose any point B not on l. Construct the perpendicular from B to l.
3. Draw a line segment AB. Construct the perpendicular bisector of AB.

(B)

4. Draw any line segment and divide it into four equal parts.
5. Construct each angle.
   a) 45° b) 135° c) 22.5° d) 75° e) 67.5°
6. Triangle ABC has vertices at A(7, 9), B(1, 1), and C(11, 3).
   a) Draw $\triangle ABC$ on a grid.
   b) Construct P, the midpoint of AB, and Q, the midpoint of AC.
   c) Compare the lengths of PQ and BC.
   d) Compare the measures of $\angle APQ$ and $\angle ABC$.
7. Construct $\triangle PQR$ in which $\angle Q = 90°$, PQ = 5.0 cm, and QR = 5.0 cm. Measure $\angle P$ and $\angle R$.
8. a) Construct $\triangle ABC$ with BC = 8.0 cm, $\angle B = 45°$, and $\angle C = 30°$.
   b) Construct the midpoint M of BC. Draw AM.
   c) Measure. i) AM ii) $\angle BAC$ iii) $\angle AMC$
9. a) Construct $\triangle XYZ$ such that XY = 4.0 cm, YZ = 5.0 cm, and XZ = 6.0 cm.
   b) Measure each angle of $\triangle XYZ$.
   c) Construct a line through X perpendicular to YZ.
   d) Construct the bisector of $\angle Y$.

10. a) Draw $\triangle ABC$ with $AB = 7.5$ cm, $\angle B = 60°$, and $BC = 10.0$ cm.
    b) Construct the perpendicular from A to BC, and measure its length.
    c) Construct the bisector of $\angle C$, and extend it to meet AB at K. Measure the length of segment CK.
    d) Construct the midpoint M of AC. Draw segment MB and measure its length.

11. Construct a regular octagon with sides of length 4.0 cm.

12. Mark two points A and B on your paper. Construct a circle having AB as a diameter.

13. Draw a line l. Choose any point P not on l. Construct a line through P parallel to l.

14. Which of the constructions described in *Sections 12-1* and *12-2* can be done with ruler and compasses, with the compasses fixed at one setting?

Ⓒ

15. Mark two points A and B on your paper. Construct each square.
    a) A and B are the endpoints of one side.
    b) A and B are the endpoints of a diagonal.
    c) A and B are the midpoints of two opposite sides.
    d) A and B are the midpoints of two adjacent sides.

16. Construct a perpendicular at a point P on a line, using a plastic triangle. Can you do this without placing the right angle of the triangle at P?

17. Explain some of the constructions described on pages 456 and 457.

INVESTIGATE

1. Draw any line segment AB, and construct its perpendicular bisector.
2. Locate any point P on the bisector of AB. Join PA and PB. Measure the lengths of the segments PA and PB.
3. Repeat *Question 2* for other points on the perpendicular bisector of AB.
4. State a probable conclusion about any point on the perpendicular bisector of a line segment.

INVESTIGATE

1. Draw around a circular object such as a jar lid. Locate the centre of the circle. Can you find more than one way to do this?
2. The photograph on page 456 shows a *centre finder* used by carpenters to locate the centre of round stock.
    a) How is the centre finder used?
    b) Explain why it works.

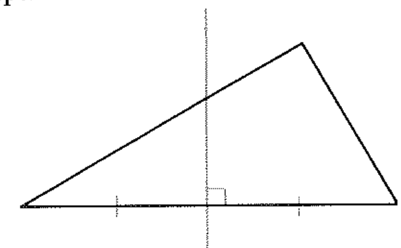
INVESTIGATE

Lines, Triangles, and Circles

The diagrams on these two pages illustrate some particular ways in which lines and triangles intersect.

Investigating Perpendicular Bisectors

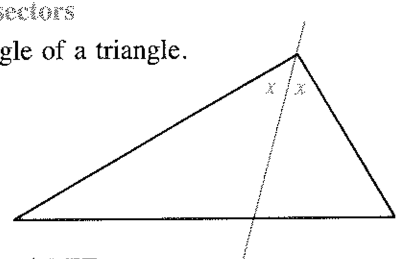
A line may be the perpendicular bisector of a side of a triangle.



1. Draw a large scalene $\triangle ABC$.
2. Construct the perpendicular bisector of each side. What do you notice?
3. Let O be the point of intersection of the perpendicular bisectors. Place the compasses point on O and draw a circle which passes through the three vertices of the triangle. We call this circle the *circumcircle* of $\triangle ABC$. O is called the *circumcentre*.
4. State a probable conclusion about the perpendicular bisectors of the sides of any triangle.

Investigating Angle Bisectors

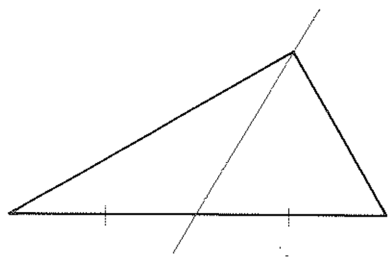
A line may bisect an angle of a triangle.



1. Draw a large scalene $\triangle DEF$.
2. Construct the bisector of each angle. What do you notice?
3. Let I be the point of intersection of the angle bisectors. Place the compasses point on I and draw a circle which touches the three sides of the triangle. We call this circle the *incircle* of $\triangle DEF$. I is called the *incentre*.
4. State a probable conclusion about the angle bisectors of any triangle.

Investigating Medians

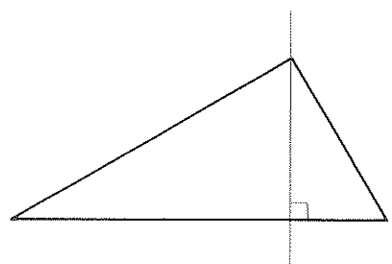
A line may pass through a vertex of a triangle and the midpoint of the opposite side. The line segment joining the vertex to the midpoint is called a *median* of the triangle.



1. Draw a large scalene △PQR on cardboard.
2. Construct the three medians of the triangle. What do you notice?
3. Carefully cut out the triangle.
   a) Try to balance the triangle on the edge of a ruler, with one of the medians on the ruler. Do this with each of the three medians.
   b) Let G be the point of intersection of the three medians. Try to balance the triangle with a pencil point at G. G is called the *centroid*, or *centre of gravity* of the triangle.
4. State a probable conclusion about the medians of any triangle.

Investigating Altitudes

A line may pass through a vertex of a triangle and be perpendicular to the opposite side. The line segment joining the vertex to the opposite side is called an *altitude* of the triangle.



1. Draw a large scalene acute △XYZ.
2. Construct the three altitudes of the triangle. What do you notice?
3. a) Draw examples of triangles to show that an altitude may be:
   i) a side of the triangle ii) outside the triangle.

   b) Repeat the investigation for these triangles.
4. Let H be the point of intersection of the altitudes. H is called the *orthocentre* of the triangle. State a probable conclusion about the altitudes of any triangle.

12-3 CONSTRUCTING CIRCLES THROUGH SETS OF POINTS

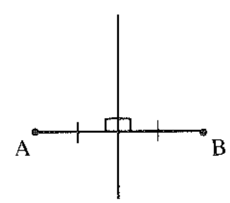
Since ancient times, mathematicians have investigated the circumstances under which circles can be drawn through various numbers of points.

Two Points

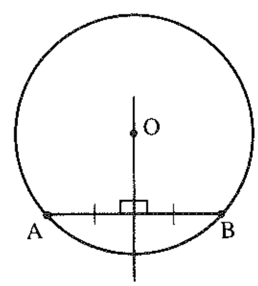
Let A and B be any two distinct points. How can we draw a circle through A and B?

In a previous *INVESTIGATE* you may have discovered that any point on the perpendicular bisector of a line segment is equidistant from its endpoints. Hence, any point on the perpendicular bisector of the segment AB can be used as the centre of a circle passing through A and B.

Step 1. Construct the perpendicular bisector of AB.



Step 2. Choose any point on the perpendicular bisector as a centre. Draw a circle through A and B.

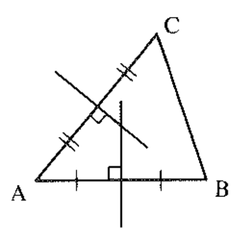


Three Points

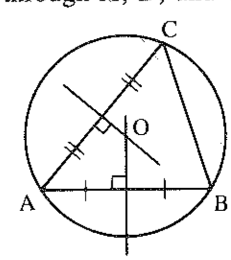
Let A, B, and C be three distinct points. How can we draw a circle through A, B, and C?

In a previous *INVESTIGATE* you may have discovered that a circle can be drawn through the three vertices of any triangle. The centre of the circle is the point of intersection of the perpendicular bisectors of the sides of the triangle.

Step 1. Draw $\triangle ABC$ and construct the perpendicular bisectors of any two sides of the triangle (using any method).

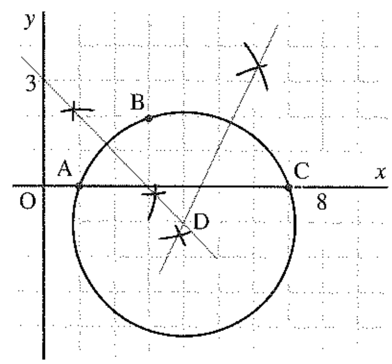


Step 2. Using the point of intersection of the perpendicular bisectors as a centre, draw a circle through A, B, and C.



Example Construct the circle which passes through A(1, 0), B(3, 2), and C(7, 0).

Solution. Plot the points on a grid. Construct the perpendicular bisectors of AB and BC. Label their point of intersection D. Place the compasses point on D and draw a circle to pass through A, B, and C.



EXERCISES 12-3

Ⓐ

1. Draw x- and y-axes on a grid.
   a) Construct a circle C_1 with centre (0,0) and diameter 8 units.
   b) Construct a circle C_2 with centre (4,0) and radius 4 units.
   c) Name the coordinates of the two points at which C_1 and C_2 intersect.

2. On a grid, construct a circle with centre (3,2) which passes through the point (5,5). Measure the diameter of the circle.

Ⓑ

3. Construct a circle which passes through each set of points.
   a) A(3,0), B(5,5), C(8,3)
   b) D(2, −3), O(0,0), E(−4, −3)
   c) F(1,3), G(−5,2), H(1, −3)

4. a) Can you construct a circle through the points R(1, −1), P(0, −4), and S(2,2)? Explain your answer.
   b) Under what conditions can a circle be constructed through three given points?

5. If any three distinct points are given, is it always possible to draw a circle which passes through them? Give examples to illustrate your answer.

6. If more than three distinct points are given, is it always possible to draw a circle which passes through them? Give an explanation, or draw some diagrams to illustrate your answer.

7. Construct △XYZ with base XY = 4 cm and altitude ZM = 3 cm. How many different triangles can be drawn?

8. Construct △XYZ with base XY = 4 cm and median ZN = 3 cm. How many different triangles can be drawn?

9. When would the triangle in *Exercise 7* and the triangle in *Exercise 8* be the same triangle?

Ⓒ

10. a) Draw $\triangle ABC$ with $AB = 11.5$ cm, $BC = 13.0$ cm, and $AC = 9.0$ cm.
    b) Construct the perpendicular bisector of each side and label the circumcentre O.
    c) Construct the three altitudes and label the orthocentre H.
    d) Construct the three medians and label the centroid G.
    e) If you have worked carefully, you should be able to draw a line through O, H, and G. It is called the *Euler line*.
    f) Measure segments OH, HG, and OG. How are the lengths related?
    g) Determine whether the incentre lies on the Euler line.

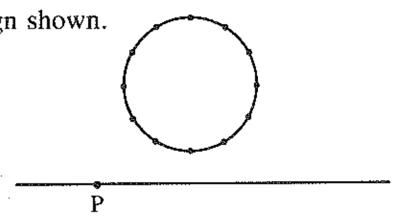
11. An altitude, a median, and an angle bisector are drawn from the same vertex of a scalene triangle and extended to meet the opposite side. Which of the three line segments divides the triangle into two equal areas?

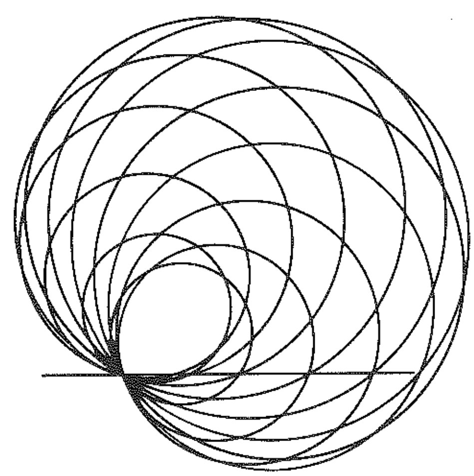
12. An interesting series of constructions leads to a circle that can be drawn through *nine* seemingly unrelated points.
    a) Draw a large scalene $\triangle ABC$ with sides longer than 20 cm.
    b) Construct the midpoints of AB, BC, and CA, and label them as P_1, P_2, and P_3 respectively.
    c) Construct the orthocentre of $\triangle ABC$ and label it H. Mark P_4, P_5, and P_6, the points where the altitudes meet AB, BC, and CA, respectively.
    d) Label the midpoints of HA, HB, and HC as P_7, P_8, and P_9, respectively.
    e) Construct the circumcentre of $\triangle ABC$ and label it O.
    f) Label the midpoint of OH as F. Using F as the centre, draw the circle which passes through P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, and P_9. This is called the *nine-point circle* of $\triangle ABC$.
    g) Determine whether the centre of the nine-point circle lies on the Euler line (see *Exercise 10*).

INVESTIGATE

Circle Patterns

Draw a circle and a line with point P, as shown. Use a protractor to mark several equally-spaced points on the circle. Using each point as a centre, draw a circle through P. The result should look like the design shown.



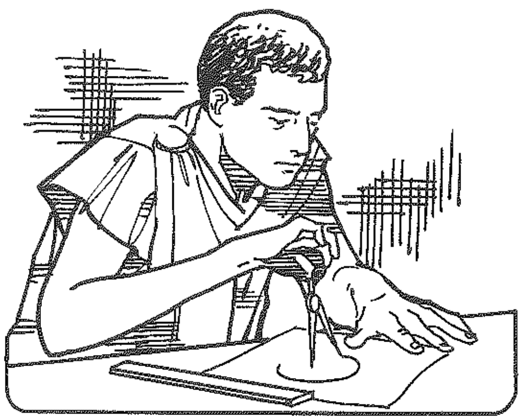


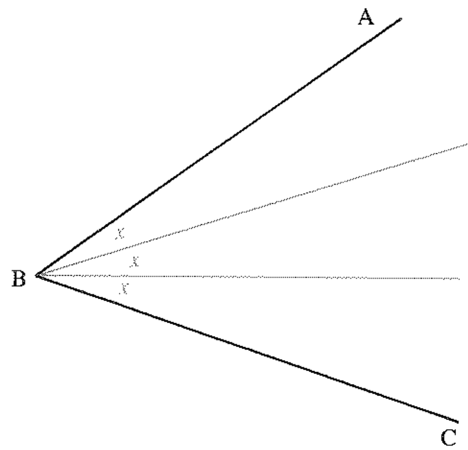
THE MATHEMATICAL MIND

A Problem that took Thousands of Years to Solve

1
The ancient Greeks established rules for constructing geometric figures. Since they considered the line and the circle to be the basic figures, only compasses and an unmarked straightedge could be used.

2
One of the constructions, which the Greek mathematicians attempted, was to *trisect* any given angle using compasses and a straightedge.





3
Since the 5th century B.C., mathematicians have tried to trisect any given angle using compasses and a straightedge. However, none has succeeded.

4
Finally, in 1837, P.L. Wantzel *proved* that such a construction is impossible! He did this by showing that a 20° angle cannot be constructed using compasses and a straightedge. This means that a 60° angle cannot be trisected.

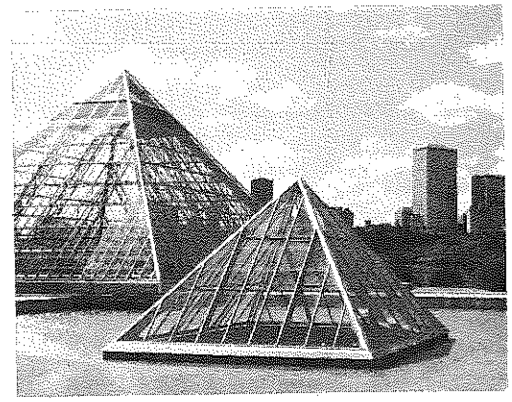
Certain angles can be trisected using compasses and a straightedge, but there is no general method that works for all angles. If the restriction on compasses and a straightedge is removed, an angle can be trisected using a variety of other instruments.

QUESTIONS

For each of these angles

i) 180° ii) 90° iii) 30° iv) 45° v) 120°

a) Construct the angle using compasses and a straightedge.
b) Determine if it is possible to trisect it using compasses and a straightedge.
c) Trisect it if you can.



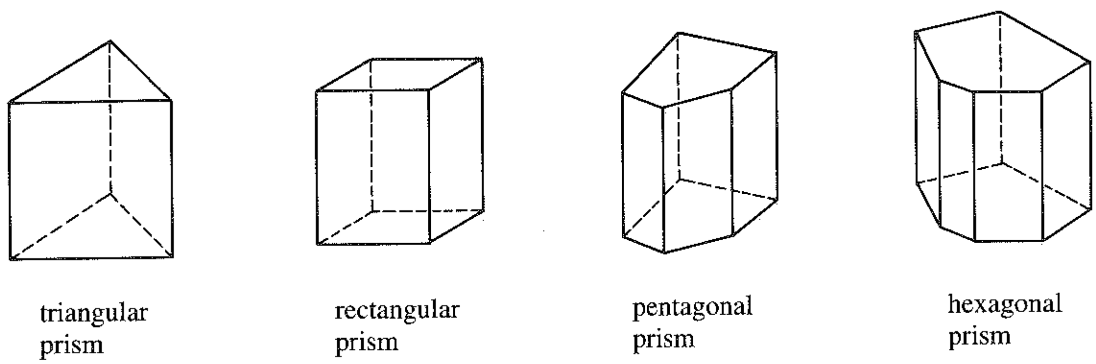
12-4 SURFACE AREAS OF PRISMS AND PYRAMIDS

We see many examples of prisms and pyramids.

When the Earth's minerals are analyzed, they are found to contain crystals of various kinds. The crystals have flat (rather than curved) faces. They are examples of solids called prisms.

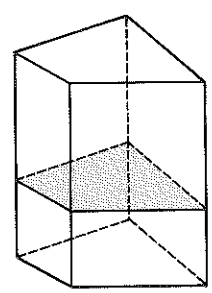
The Muttart Conservatory in Edmonton consists of two pyramidal buildings. The faces of a pyramid (other than the base) all meet at the top of the pyramid.

Here are four types of prisms.

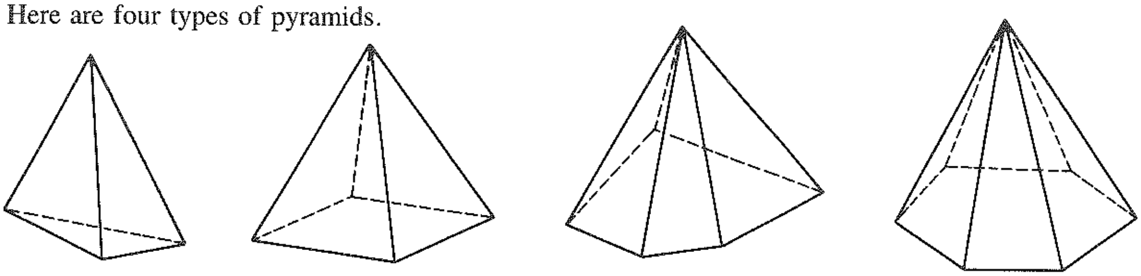


A *prism* is a solid with two congruent and parallel faces called *bases*. The other faces of the prism are parallelograms.

A plane parallel to the bases of a prism intersects the prism in a figure that is *congruent* to the bases.



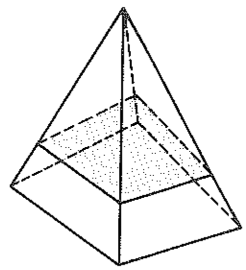
Here are four types of pyramids.



triangular pyramid | rectangular pyramid | pentagonal pyramid | hexagonal pyramid

A *pyramid* is a solid with a polygonal base. The other faces are triangles with a common vertex.

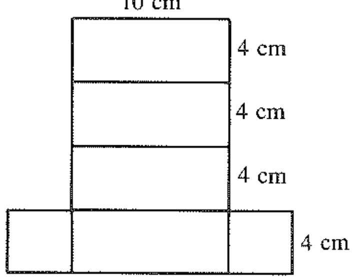
A plane parallel to the base of a pyramid intersects the pyramid in a figure that is *similar* to the base.



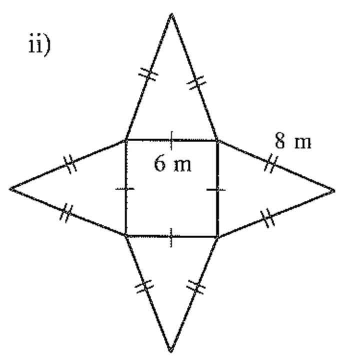
The faces of a prism or a pyramid can be arranged in a pattern called a *net*. Such a pattern could be cut from cardboard, and folded to form the solid. Since the area of a net does not change when it is folded into the solid, we can calculate the area of the net to determine the *surface area* of the solid.

Example 1. a) Identify the solid formed by each net.
b) Calculate the surface area of each solid.

i)



ii)



Solution. i) a) The net will fold to form a rectangular prism.

b) The net consists of four rectangles each with area 40 cm^2 and two squares each with area 16 cm^2.

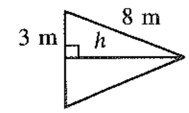
$$\text{Area of net} = 4(40) + 2(16) = 192$$

The surface area of the prism is 192 cm^2.

ii) a) The net will fold to form a square-based pyramid.

b) The net consists of a square with area 36 m^2 and four triangles with sides 8 m, 8 m, and 6 m.
Before finding the area of each triangle, we use the Pythagorean Theorem to calculate its height h.

$$h = \sqrt{8^2 - 3^2} = \sqrt{55}$$



$$\text{Area of each triangle} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(6)(\sqrt{55}) = 3\sqrt{55}$$

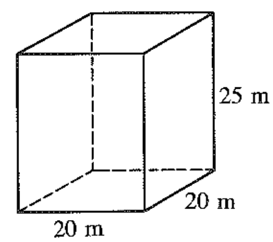
$$\text{Area of net} = 4(3\sqrt{55}) + 36 \doteq 124.99$$

The surface area of the pyramid is about 125 m^2.

Example 2. A nation plans to construct, for a World's Fair, a glass pavilion on a square base measuring 20 m along each side. Plan A calls for a rectangular prism 25 m high. Plan B calls for a pyramid 50 m high. Which plan would use less glass?

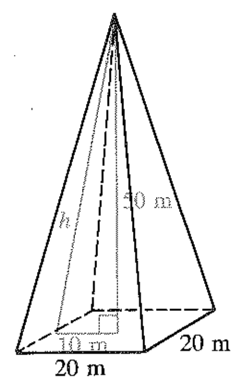
Solution. Draw a diagram of each pavilion.

Plan A



The surface area, in square metres, of the rectangular prism (excluding the base) is $4(20 \times 25) + (20 \times 20)$, or 2400.

Plan B



To find the area of the triangular faces of the pyramid, we must first calculate their height h.
From the diagram, we see that h is the hypotenuse of a right triangle with sides 50 m and 10 m.

$$h = \sqrt{50^2 + 10^2} = \sqrt{2600}$$

$$\text{Area of each triangular face} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(20)(\sqrt{2600}) = 10\sqrt{2600}$$

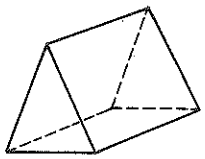
The surface area, in square metres, of the glass pyramid (excluding the base) is $4(10)\sqrt{2600}$, or about 2040.
The pyramid uses less glass than the prism.

EXERCISES 12-4

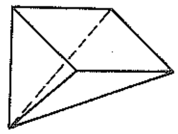
(A)

1. Which solids are prisms? Which are pyramids?

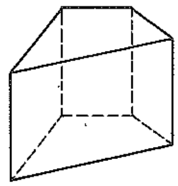
a)



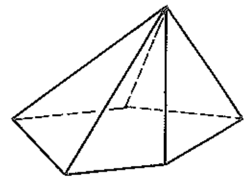
b)



c)

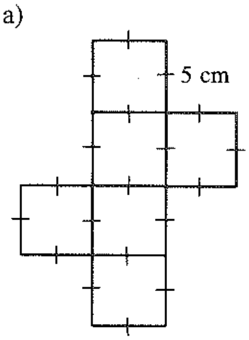


d)

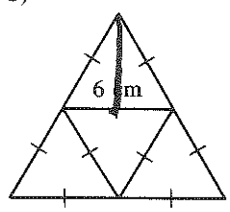


2. Find the area of each net.

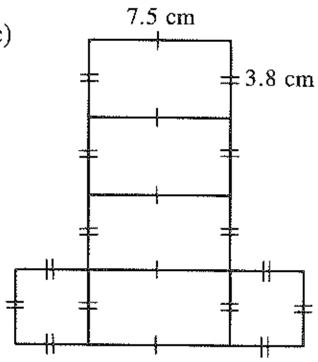
a)



b)

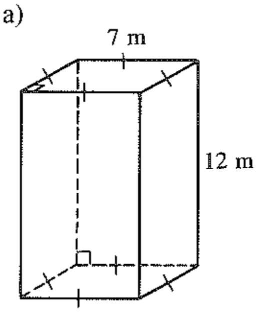


c)

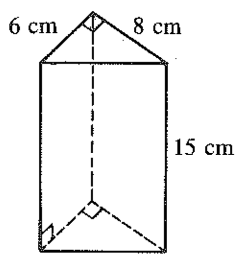


3. Find each surface area.

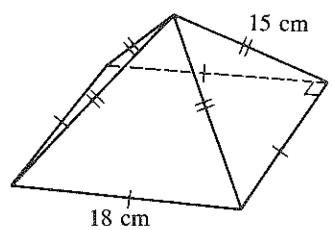
a)



b)



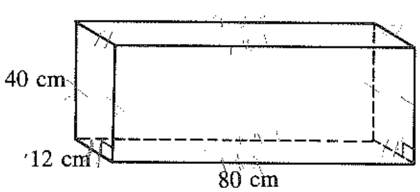
c)



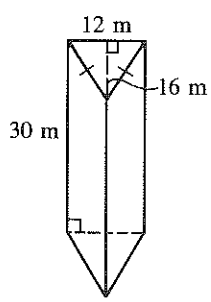
(B)

4. Draw a net and find each surface area.

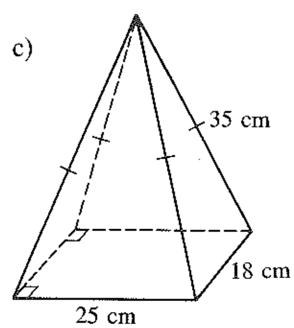
a)



b)

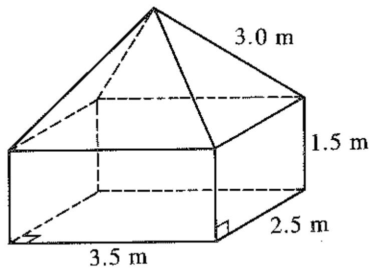


c)

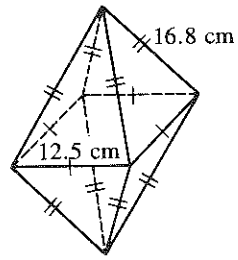


5. Find each surface area.

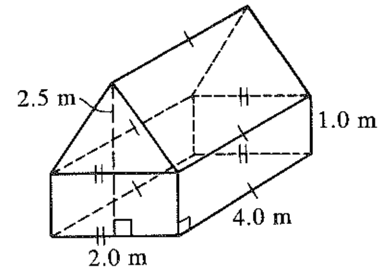
a)



b)



c)



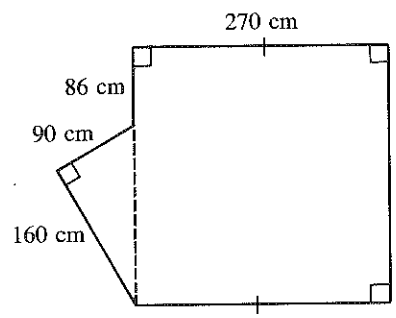
6. A hexagonal prism, with a height of 27 cm, has a base consisting of six equilateral triangles with edges 32 mm long.
   a) Draw the net. b) Find the total surface area.

7. If you know how many sides there are in the base of a prism, how could you determine the number of:
   a) faces b) vertices c) edges?

8. If you know how many sides there are in the base of a pyramid, how could you determine the number of:
   a) faces b) vertices c) edges?

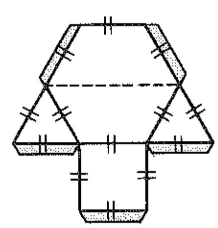
Ⓒ

9. Susan is going to redecorate her bedroom, which has the floor plan shown. The ceiling is 2.4 m high. The paint costs $0.70/m^2$, the wallpaper costs $9.50/m^2$, and the carpet costs $32.50/m^2$. How much will it cost:
   a) to carpet the floor
   b) to paint the walls and the ceiling
   c) to paper the walls and paint the ceiling?



INVESTIGATE

1. Construct two copies of this net, and use tape to make two solids with them.
2. Fit these solids together to make a pyramid.
3. Calculate the surface area of the pyramid.



INVESTIGATE

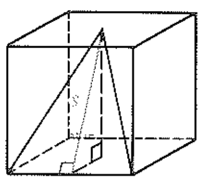
Constructing a Pyramid

To construct a pyramid and investigate its volume you will need these materials: a rectangular box, some pieces of cardboard, and some tape.

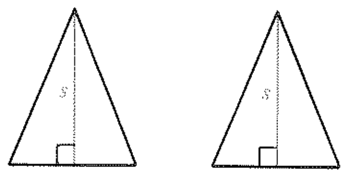
Use the steps below to construct a pyramid which will fit inside the box.

Step 1. Measure the length, the width, and the height of the box. Record the results.

Step 2. Use the Pythagorean Theorem to calculate the slant height s of a face whose base is the length of the box.



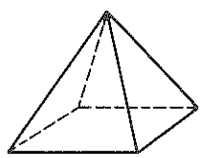
Step 3. Construct two triangles with this height and with bases equal to the length of the box.



Step 4. Repeat Steps 2 and 3 for a face whose base is the width of the box.

Step 5. Construct a rectangle which is congruent to the base of the box.

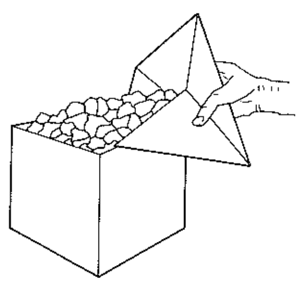
Step 6. Tape the four triangles and the rectangle together to form a pyramid.



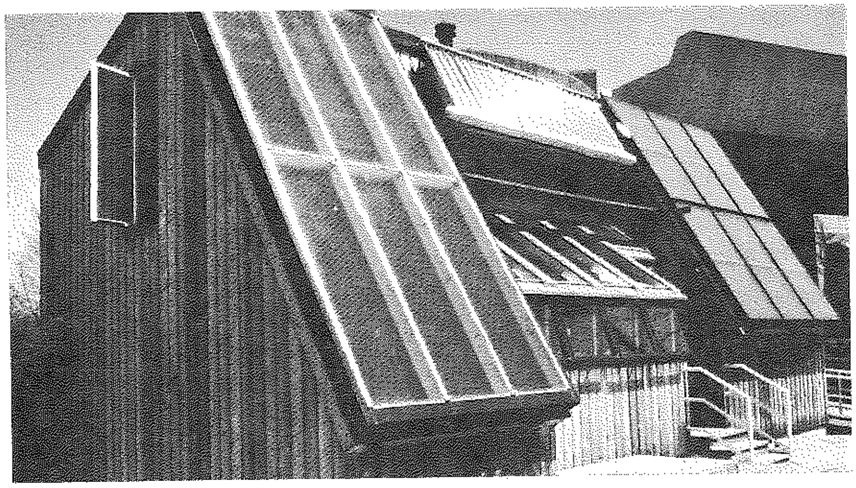
Estimating the Volume of a Pyramid

Use the pyramid you constructed above, with the base removed. Without the base, the pyramid will not be rigid, and it may be necessary to reinforce it with tape.

1. Suppose the pyramid were filled with sand, and then emptied into the box. Estimate how many times this could be done until the box is full.
2. Check your estimate. Instead of using sand, use some other material such as loose styrofoam pieces which are often used as packing material.



3. What probable conclusion can you make about the volume of a pyramid compared with the volume of a rectangular prism which has the same height and the same base?

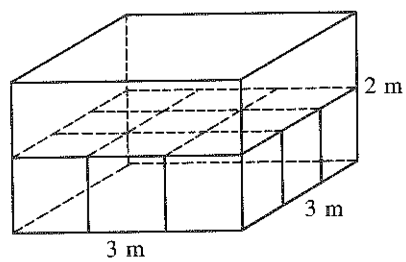


12-5 VOLUMES OF PRISMS AND PYRAMIDS

To conserve energy, some houses are heated by solar power that is collected in the daytime in large collectors on the roof. To store the heat for later use, air in the collectors is pumped into a large container of rocks buried under the house. The volume of the container is an important consideration since it must be large enough to heat the house properly.

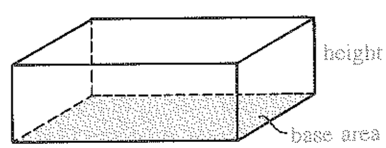
Volume of a Prism

The rock container shown is a rectangular prism measuring 3 m by 3 m by 2 m. To determine the volume of the container, think of filling it with unit cubes in layers.

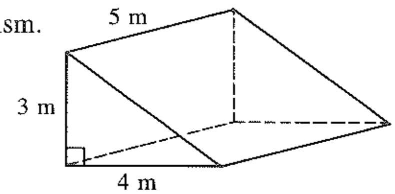


The area of the base is the number of cubes which cover it: 3×3, or 9. If this is multiplied by the number of layers, the result is the number of cubes which fill the container.
Hence, the volume of the container is the base area multiplied by the height: 9×2, or 18.
The rock container has a volume of 18 m^3.

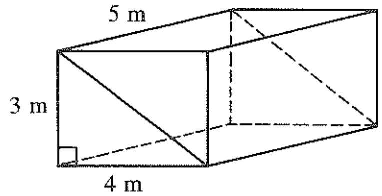
The volume V of a rectangular prism is given by this formula.
V = (base area)(height)



Example 1. Find the volume of this triangular prism.



Solution. A rectangular prism, measuring 3 m by 4 m by 5 m, can be divided into two triangular prisms congruent to the one given. This suggests that the volume of the triangular prism is one-half of the volume of the rectangular prism.

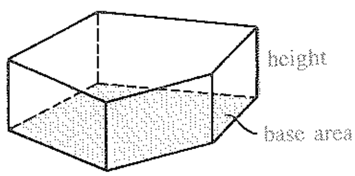


$$V = \frac{1}{2}(3)(4)(5)$$
$$= 30$$

The volume of the triangular prism is 30 m^3.

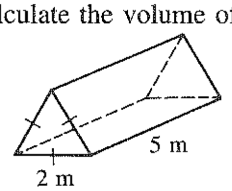
In *Example 1*, the expression $\frac{1}{2}(3)(4)$ represents the area of the base of the triangular prism. Since the volume was obtained by multiplying this expression by the height, this suggests a general formula for the volume of any prism.

The volume V of any prism is given by this formula.
V = (base area)(height)

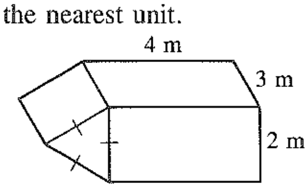


Example 2. Calculate the volume of each solid, to the nearest unit.

a)

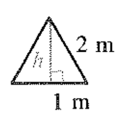


b)



Solution. a) The solid is a triangular prism, with its base an equilateral triangle. To find the area of the base, we use the Pythagorean Theorem to calculate the height of the triangle.

$$h = \sqrt{2^2 - 1^2}$$
$$= \sqrt{3}$$



Area of the triangular base $= \frac{1}{2}(2)(\sqrt{3})$
$= \sqrt{3}$

Volume of the prism, V = (base area)(height)
$= (\sqrt{3})(5)$
$\doteq 8.7$

The volume of the prism is about 9 m^3.

b) The solid comprises a rectangular prism and a triangular prism. The rectangular prism has a base measuring 2 m by 3 m.
Volume of the rectangular prism, V = (base area)(height)
$= (2 \times 3)(4)$
$= 24$

The triangular prism has a base area equal to that of the prism in part a).
Volume of the triangular prism, V = (base area)(height)
$= (\sqrt{3})(3)$
$= 3\sqrt{3}$

Total volume $= 24 + 3\sqrt{3}$
$\doteq 29.2$

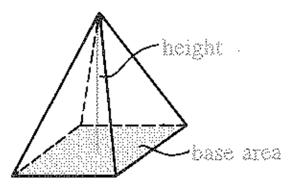
The volume of the solid is about 29 m^3.

Volume of a Pyramid

In a preceding *INVESTIGATE* you should have found that there is a simple relationship between the volumes of certain pyramids and prisms. If a pyramid and a prism have the same base and the same height, then the volume of the pyramid is one-third of the volume of the prism.

The volume V of a pyramid is given by this formula.

$$V = \frac{1}{3}(\text{base area})(\text{height})$$

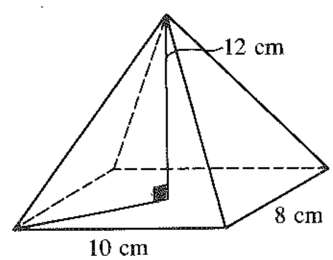


Example 3. Find the volume of the pyramid.

Solution. The base is a rectangle with area 80 cm^2.
Hence, the volume of the pyramid is

$$V = \frac{1}{3}(\text{base area})(\text{height})$$
$$= \frac{1}{3}(80)(12)$$
$$= 320$$

The volume of the pyramid is 320 cm^3.

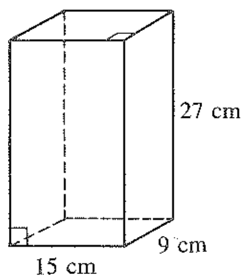


EXERCISES 12-5

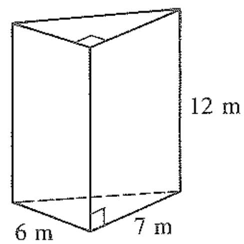
(A)

1. Calculate each volume to the nearest unit.

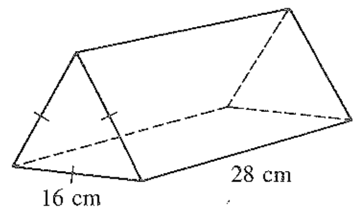
a)



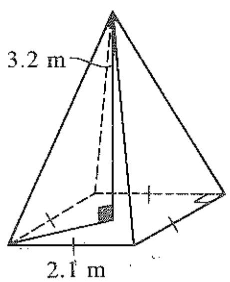
b)



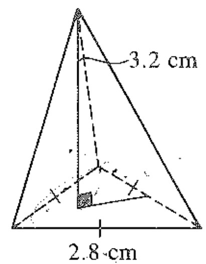
c)



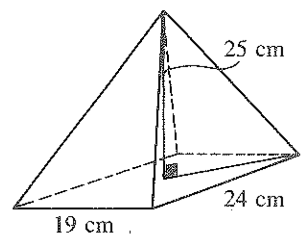
d)



e)

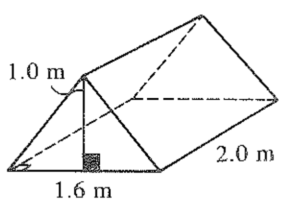


f)



2. Calculate the volume of space enclosed by each tent, to the nearest unit.

a)



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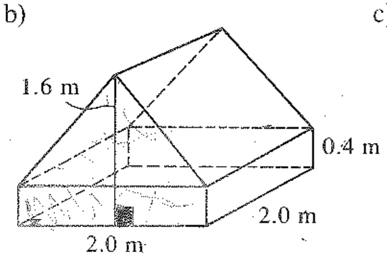
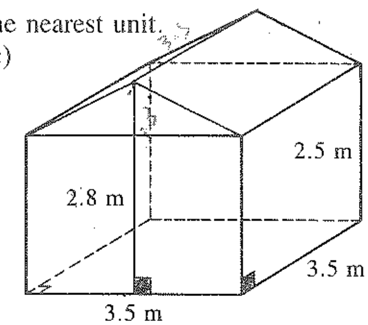


c)



3. Shipping companies often use *CAST* containers to reduce handling costs. One such container measures 3.2 m by 2.6 m by 2.2 m. What volume of goods can be shipped in it?

4. A granary measures 1.8 m by 2.6 m by 2.2 m. What is its capacity?

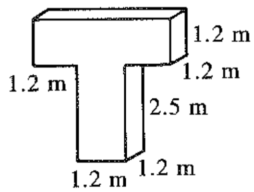
(B)

5. The bucket on a front-end loader measures 1.2 m by 2.3 m by 1.8 m. The bin of a dump truck measures 4.2 m by 2.5 m by 1.9 m. How many bucket loads will it take to fill the truck?

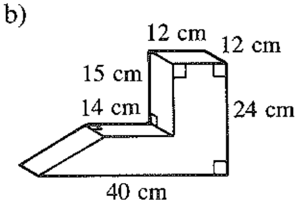
6. Meridian Frozen Foods sells its products in boxes measuring 38 cm by 28 cm by 7.0 cm. How many of these boxes can be packed into a freezer space measuring 3.50 m by 1.52 m by 0.84 m?

7. Calculate each volume to the nearest unit.

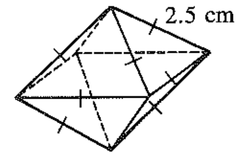
a)



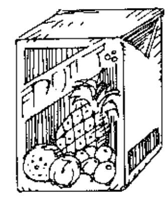
b)



c)

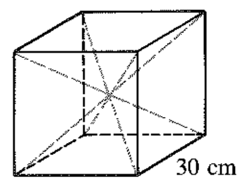


8. Juice drinks come in 250 mL containers shaped like a rectangular prism.
   a) Measure a container like this one.
   b) Calculate its volume in cubic centimetres.
   c) Does the result confirm that the container holds 250 mL of juice?



9. An aquarium measures 60 cm by 30 cm by 30 cm. If 10 L of water are poured in, how high will the water level rise?

10. A cube is divided into six congruent pyramids. The base of each pyramid is a face of the cube. If the edges of the cube are 30 cm long, what is the volume of each pyramid?



Ⓒ

11. In *Exercise 6*, the food boxes filled the freezer space exactly; there was no wasted space. Suppose the freezer space were 1.11 m by 1.40 m by 2.10 m.
    a) How many boxes could be packed into the freezer?
    b) How much wasted space would there be?

12. The surface area of a cube is 100 cm^2. Calculate its volume to three significant digits.

13. The Great Pyramid in Egypt has a square base of 230 m. Its original height was 147 m; now its height is 146 m.
    a) Find the volume of stone that was used to build the pyramid.
    b) About 2.3×10^6 blocks of stone were used to build the pyramid. Find the volume of one block.
    c) One block of stone has a mass of about 2.5 t. Find the mass of stone in the pyramid today.

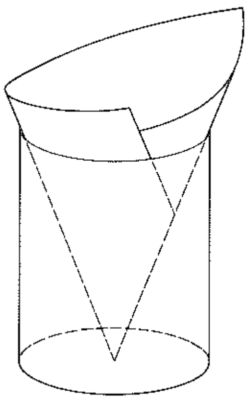
14. During a snowstorm, 8 cm of snow fell. When snow melts, the volume of the water is about $\frac{1}{10}$ of the volume of the snow.
    a) What volume of snow fell on a lawn measuring 20 m by 15 m?
    b) If the water from the melted snow were collected, how many aquariums like the one in *Exercise 9* would it fill?

INVESTIGATE

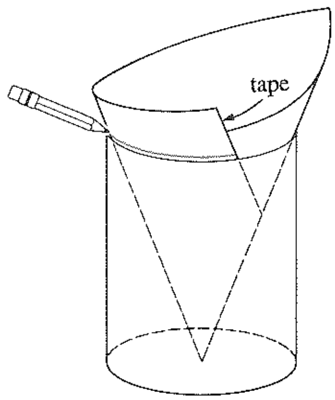
The Volume of a Cone

To investigate the volume of a cone, follow these steps. You will need an empty tin can such as a soup can, a piece of paper, and some tape.

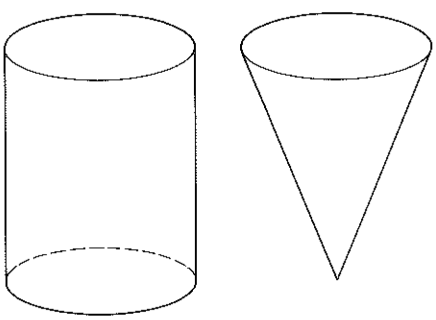
Step 1.
Roll the piece of paper into the shape of a cone, and adjust it such that it just fits in the tin can.



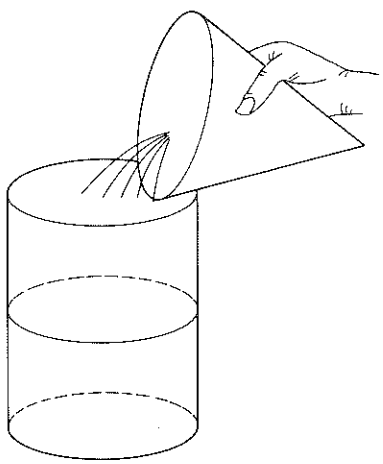
Step 2.
Secure the cone with tape so that it does not unroll. Mark the rim of the can on the cone, and cut off the excess.



Step 3.
Suppose the cone were filled with water, and then emptied into the can. Estimate how many times this could be done until the can is full.



Step 4.
Check your estimate with water.

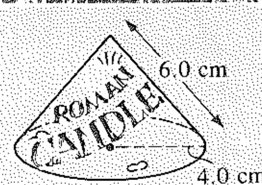


What probable conclusion can you make about the volume of a cone compared with the volume of a cylinder which has the same height and the same base radius?

PROBLEM SOLVING

Use Spatial Visualization

A firecracker is in the shape of a cone. The base radius is 4.0 cm, and the slant height is 6.0 cm. What area of paper is needed to cover the curved surface of the firecracker?



Understand the problem

- Can the curved surface be covered with one piece of paper?
- What does *slant height* mean?

Think of a strategy

- Visualize cutting the curved surface and unrolling it.

Carry out the strategy

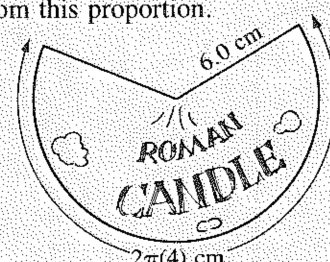
- When the curved surface is unrolled, it forms part of a circle. The radius of the circle is the slant height of the cone. The area of the curved surface of the cone is equal to the area of the part circle.
- We can find the area of the curved surface from this proportion.

$$\frac{\text{Area of cone}}{\text{Area of circle}} = \frac{\text{Circumference of cone}}{\text{Circumference of circle}}$$

$$\frac{\text{Area of cone}}{\pi(6)^2} = \frac{2\pi(4)}{2\pi(6)}$$

$$\text{Area of cone} = 36\pi\left(\frac{4}{6}\right)$$

$$\doteq 75.3982$$



The area of the curved surface of the cone is about 75 cm^2.

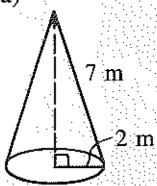
Look back

- What is the total surface area of the cone (including the base)?

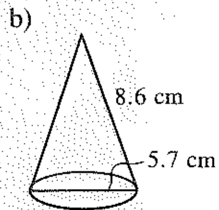
Solve each problem

1. Find the area of the curved surface of each cone.

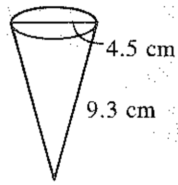
a)



b)



c)

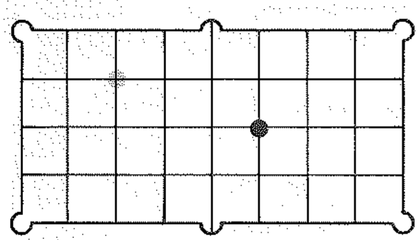


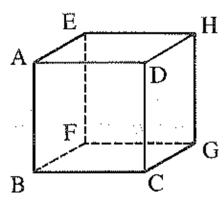
2. Find the total surface area of the cones in *Question 1*.

3. A tepee has the shape of a cone with a slant height of 2.0 m, and a base diameter of 3.0 m. Find the area of the material needed to make the tepee.

4. For Hallowe'en, a witch's hat is made by stapling together the straight edges of a quarter of a circle of radius 30 cm.
   a) How high is the hat?
   b) What is the radius of the base of the hat?

5. There are two balls on a billiard table (below left). Where should the black ball be aimed so that after one bounce it hits the colored ball?

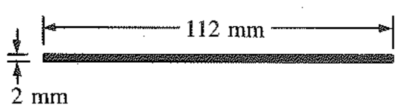




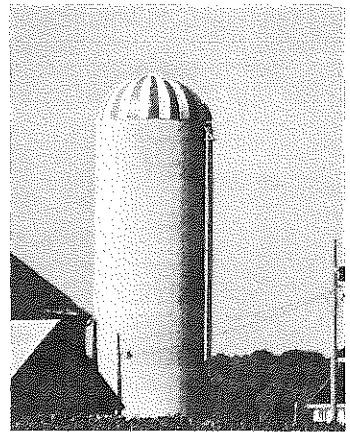
6. For the cube shown (above right), describe the figure formed by these points.
   a) DCGH   b) ADGF   c) ADCG

7. You have an empty tin can with no markings on it. How can you put the right amount of water in it so that it is exactly half full? No other container may be used.

8. A silver dollar with an inked edge leaves the trace shown when making one complete roll on paper. Find the volume of the coin, to the nearest cubic millimetre.



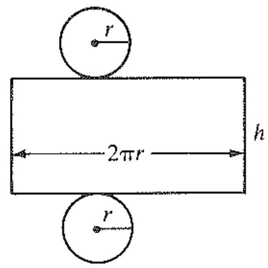
12-6 CYLINDERS AND CONES



Why are silos built with circular bases rather than square ones?

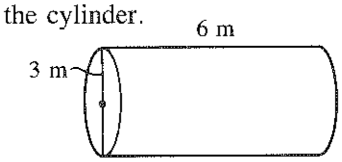
In the construction of tanks, granaries, silos, and other storage facilities, the shape of the structure is an important consideration. A cylinder, a cone, a prism, and a pyramid constructed from the same amount of material will have different storage capacities, or volumes.

Surface Area of a Cylinder

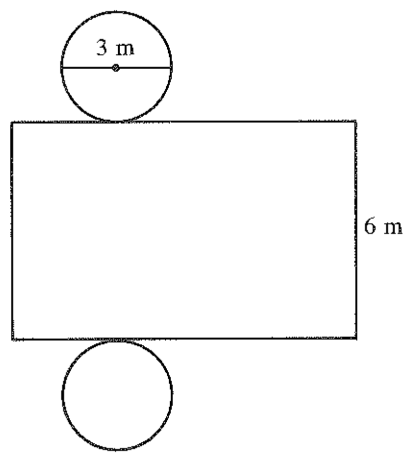


The net for a cylinder is a rectangle and two circles. The length of the rectangle is the circumference of either circle. The width of the rectangle is the height of the cylinder. The surface area of the cylinder formed from the net is the sum of the areas of the rectangle and the two circles.

Example 1. Find the surface area of the cylinder.



Solution. Draw the net of the cylinder.



Since the diameter is 3 m, the radius of each circle is 1.5 m.

Area of each circle $= \pi r^2$
$= \pi(1.5)^2$
$= 2.25\pi$

The length of the rectangle is the circumference of either circle.

Circumference of each circle $= \pi d$
$= \pi(3)$

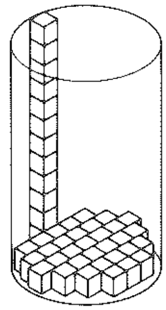
Area of the rectangle $= \pi(3)(6)$
$= 18\pi$

Area of the net $= 2(2.25\pi) + 18\pi$
$= 22.5\pi$
$\doteq 70.7$

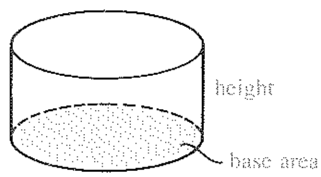
The surface area of the cylinder is about 71 m^2.

Volume of a Cylinder

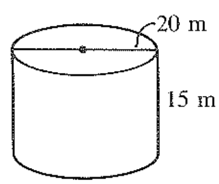
We visualize the volume of a cylinder in the same way that we visualize the volume of a prism. Imagine that the cylinder is filled with layers of unit cubes. The area of the base is the number of cubes, including part cubes, which cover it. If this area is multiplied by the number of layers, the result is the total number of unit cubes which fill the cylinder. But this is the volume of the cylinder. Hence, the volume can be found by multiplying the area of the base by the height.



The volume V of a cylinder is given by this formula.
$V =$ (base area)(height)



Example 2. Find the volume, to the nearest 100 m^3, of the storage tank with the dimensions shown.

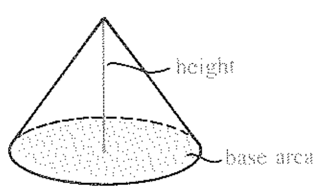


Solution. Volume of the tank, $V =$ (base area)(height)
Since base area $= \pi r^2$, $V = \pi r^2$(height)
Substitute 10 for r and 15 for height.
$V = \pi(10)^2(15)$
$\doteq 4712$
The volume of the storage tank is about 4700 m^3.

Volume of a Cone

In the preceding *INVESTIGATE* you should have found that there is a simple relationship between the volumes of certain cylinders and cones. If a cone and a cylinder have the same height and the same base radius, then the volume of the cone is one-third the volume of the cylinder.

The volume V of a cone is given by this formula.
$V = \frac{1}{3}$(base area)(height)

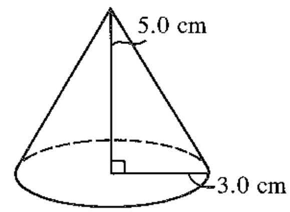


Example 3. A firecracker has the shape of a cone with a base radius of 3.0 cm and a height of 5.0 cm. Calculate the volume of the cone, to two significant digits.

Solution. Draw a diagram. The base is a circle with radius 3.0 cm.

Base area $= \pi r^2$

$= \pi(9)$



The height is 5.0 cm.

Hence, the volume of the cone is

$$V = \frac{1}{3}(\text{base area})(\text{height})$$
$$= \frac{1}{3}\pi(9)(5.0)$$
$$\doteq 47.1$$

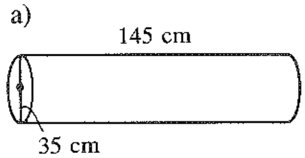
To two significant digits, the volume of the cone is 47 cm^3.

EXERCISES 12-6

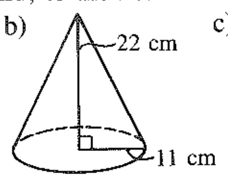
A

1. Find the volume of each solid, to the nearest unit.

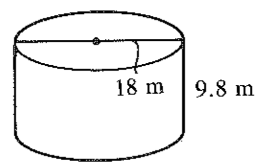
a)



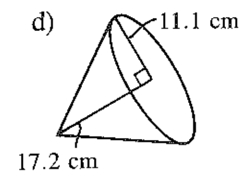
b)



c)



d)

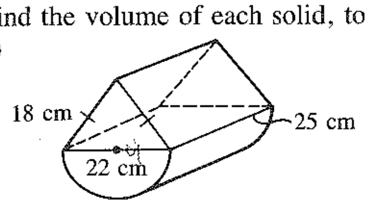


2. Find the surface area of each cylinder in *Exercise 1*.

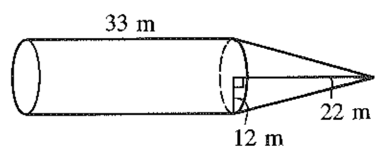
B

3. Find the volume of each solid, to the nearest unit.

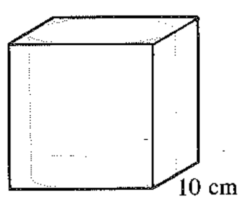
a)

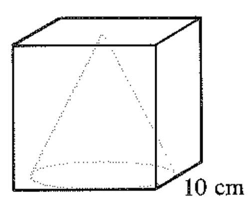


b)



4. A cylinder (below left) just fits inside a cubical box with edges 10 cm long. What is the volume of the cylinder?





5. A cone (above right) just fits inside a cubical box with edges 10 cm long. What is the volume of the cone?

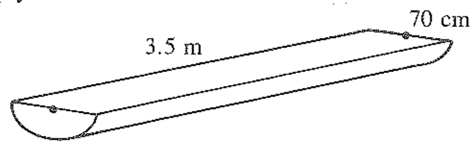
6. A square-based prism with edges 4.4 cm and a cylinder with 5.6 cm diameter have heights of 24 cm. Calculate the areas of their bases, and their volumes, to two significant digits.

7. An ice-cream cone has a diameter of 5 cm and a height of 10 cm. If it were filled with ice cream, and levelled off, how much ice cream would it contain?

8. A refinery has five cylindrical storage tanks each measuring 13.4 m in diameter and 8.7 m high. What is the total storage capacity of the refinery?

9. Tanker trucks have cylindrical tanks 18.4 m long and 2.3 m in diameter. How many truck loads would be needed to empty one of the tanks in *Exercise 8*?

10. A livestock feeder has the dimensions shown. What is the cost of the steel sheet used in its construction if steel costs $6.25/m²?



11. A square-based prism with edges 14.0 cm and a cylinder with 15.8 cm diameter have heights of 20.0 cm.
    a) Calculate the areas of their bases.
    b) Find the cost of electroplating the prism and the cylinder at 1.7¢/cm².

12. A cylindrical silo has a height of 12.5 m and a base diameter of 6.4 m. Its top is a cone with the same diameter, and height 1.2 m. Calculate the volume of the silo.

13. A cylindrical can with a base diameter of 6.4 cm is partially filled with water. When a stone is placed in the can, the water level rises 1.5 cm. Calculate the volume of the stone, to the nearest tenth of a cubic centimetre.

14. A can of paint is marked 978 mL. It has a base diameter of 10.4 cm and a height of 12.5 cm. Calculate the volume of the can, in cubic centimetres. Does the result confirm that the can's capacity is 978 mL?

15. A pipeline connects a natural-gas well to a storage depot 3.2 km away. The diameter of the pipe is 0.92 m. What volume of gas, to the nearest 1000 L, will be in the pipe?

16. Pronto gas bar sells unleaded gas at a profit of 1.6¢/L. This gas is stored underground in a cylindrical tank 2.2 m in diameter and 4.5 m long.
    a) If the average fill-up is 40 L, how many cars can be filled up?
    b) How much profit will be made?

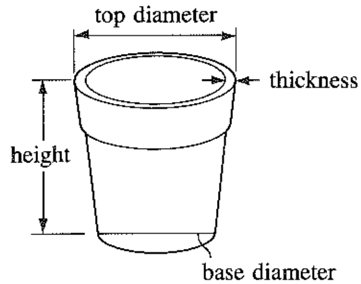
©

17. A farmer plans to build a silo 5.2 m high to hold 72 m³ of corn.
    a) If the silo were cylindrical, what would the diameter of the base be?
    b) If the silo were a square-based prism, how wide would the base be?
    c) What area of metal would be needed to make the sides of the silos in parts a) and b)?
    d) Why are silos normally built with circular bases?

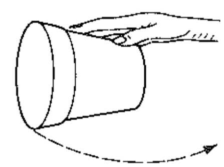
INVESTIGATE

The Styrofoam Cup

Obtain a styrofoam cup, and make the measurements shown in the diagram. Record the results for use in the investigations below.

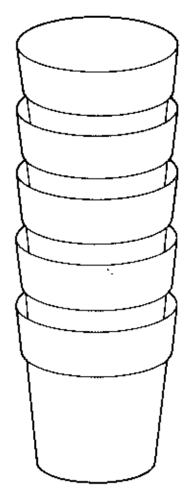


1. Place the cup on its side, and let it roll in a large circle. Do this on a large table or on the floor.
   a) Estimate the diameter of the circle.
   b) Locate the centre of the circle as accurately as you can.
   c) Measure the radius of the circle from the centre to the larger end of the styrofoam cup.



2. In *Question 1*, as the styrofoam cup traces out the circle, it rotates on its axis.
   a) Put a mark on the rim of the styrofoam cup, and let it roll around the circle. How many complete rotations does the styrofoam cup make on its axis?
   b) Confirm the result of part a) by calculation.

3. Styrofoam cups are packed by stacking them together. If you know how many cups there are, how could you determine the height of the stack?



4. The styrofoam cup has the shape of a *truncated cone*. This means that if the curved part of the cup were extended past the smaller end, a cone would result. Determine the height of this cone.

5. Calculate the capacity of the styrofoam cup, in cubic centimetres. Check by filling it with water, and measuring the volume of the water.

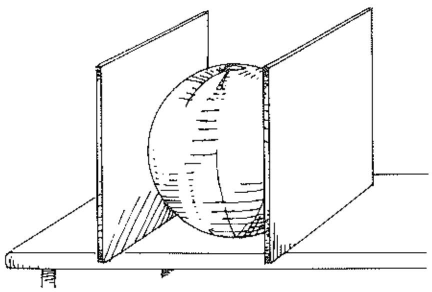
6. Find an approximation for the volume of styrofoam used to make the cup. Make any assumptions that seem reasonable.

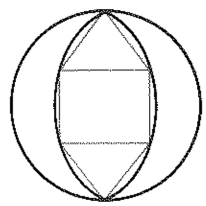
INVESTIGATE

The Surface Area of a Sphere

To investigate the surface area of a sphere, Lesley used a beach ball which was nearly spherical in shape. She carefully measured its diameter. It was approximately 37 cm.

The beach ball was divided into six congruent sections. Lesley approximated the surface area by dividing each section into two triangles and a rectangle.





She measured the dimensions of the triangles and the rectangle, and used the results to approximate the surface area of the beach ball.

Area of two triangles: $2 \times \frac{1}{2} \times 18\text{ cm} \times 21\text{ cm} = 378\text{ cm}^2$

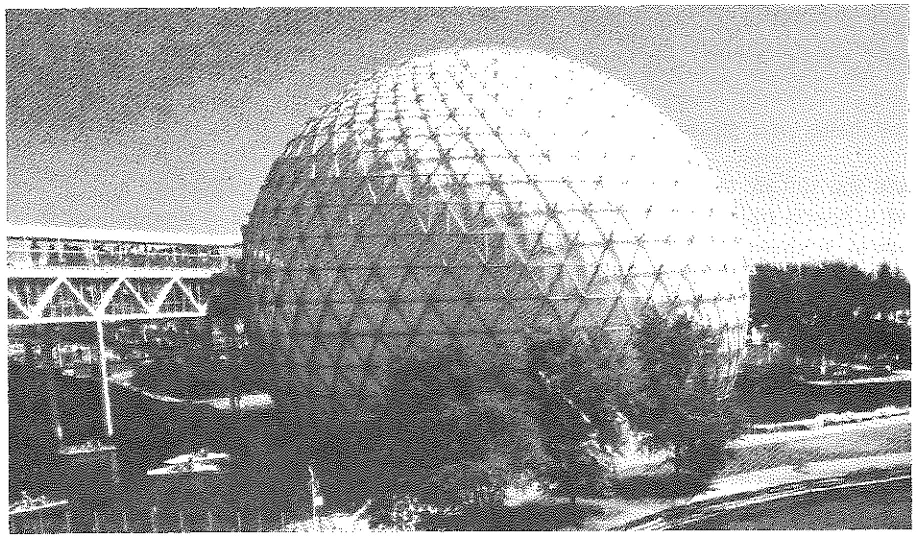
Area of rectangle: $17\text{ cm} \times 21\text{ cm} = 357\text{ cm}^2$

Total area of one section: 735 cm^2

Approximate surface area of beach ball: $6 \times 735\text{ cm}^2$, or 4410 cm^2

Then Lesley divided the surface area of the beach ball by the square of its diameter. She discovered a surprising result, and used it to form a probable conclusion about the surface area of a sphere.

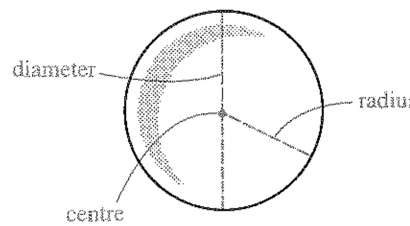
1. Complete Lesley's calculation. What did she discover?
2. a) Obtain a beach ball like the one shown and measure its diameter.
   b) Find an approximation of the surface area of the beach ball.
   c) Divide the area by the *square* of the diameter. Record the result.
3. If possible, repeat the investigation with a different beach ball, or with some other ball such as a basketball. Find the *average* value of the results in *Question 2c).*
4. A beach ball is an example of a sphere. What probable conclusion can you make about the surface area of a sphere?



12-7 SURFACE AREA AND VOLUME OF A SPHERE

The Cinesphere in Ontario Place, Toronto, has the shape of part of a sphere. How can we find the surface area and the volume of a sphere?

A *sphere* is a set of points in space which are the same distance from a fixed point, called the *centre*. A line segment joining the centre to any point on the sphere is called its *radius*. A line segment joining two points on a sphere and passing through the centre is called its *diameter*.



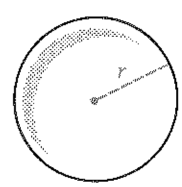
Finding the Surface Area of a Sphere

In the preceding *INVESTIGATE*, you should have found that the surface area of a sphere is slightly more than 3 times the square of the diameter. This suggests that the surface area A of a sphere with diameter d is πd^2. Since $d = 2r$, we may write

$$A = \pi(2r)^2$$
$$= 4\pi r^2$$

The surface area A of a sphere with radius r is given by this formula.

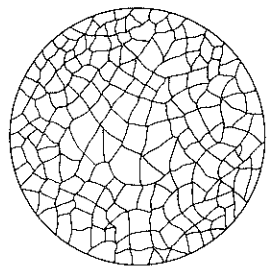
$A = 4\pi r^2$



Finding the Volume of a Sphere

We can use the formula for the surface area of a sphere to find a formula for the volume of a sphere.

Imagine that the surface of a sphere with radius r is divided into a very large number of small "polygons" with base areas A_1, A_2, A_3, and so on. Now imagine joining the vertices of these polygons to the centre of the sphere. This forms a large number of "pyramids" with heights equal to the radius of the sphere, and base areas A_1, A_2, A_3, and so on.



The volume V of the sphere is the sum of the volumes of all the pyramids. Hence,

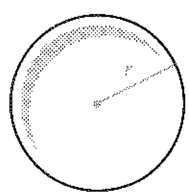
$$\begin{aligned} V &= \frac{1}{3}A_1r + \frac{1}{3}A_2r + \frac{1}{3}A_3r + \ldots \\ &= \frac{1}{3}r(A_1 + A_2 + A_3 + \ldots) \end{aligned}$$

But the expression in the brackets, $A_1 + A_2 + A_3 + \ldots$, represents the sum of the areas of the bases of all the pyramids. This is equal to the surface area of the sphere, which we know is $4\pi r^2$. Hence, we may substitute $4\pi r^2$ for the expression in the brackets. Therefore,

$$\begin{aligned} V &= \frac{1}{3}r(4\pi r^2) \\ &= \frac{4}{3}\pi r^3 \end{aligned}$$

The volume V of a sphere with radius r is given by this formula.

$$V = \frac{4}{3}\pi r^3$$



Example. The sculpture at the Japanese Embassy in Ottawa is in the shape of a sphere with a diameter of 3.2 m.

a) What is the volume of the sculpture, to the nearest cubic metre?

b) What is the surface area of the sculpture, to the nearest square metre?

Solution. The radius of the sphere is $\frac{1}{2}(3.2)$ m, or 1.6 m.

a) Substitute 1.6 for r in the formula for the volume of a sphere.

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(1.6)^3 \\ &\doteq 17.2 \end{aligned}$$

The volume of the sculpture is approximately 17 m^3.

b) Substitute 1.6 for r in the formula for the surface area of a sphere.

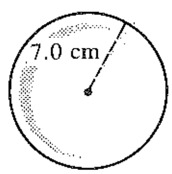
$$\begin{aligned} A &= 4\pi r^2 \\ &= 4\pi(1.6)^2 \\ &\doteq 32.2 \end{aligned}$$

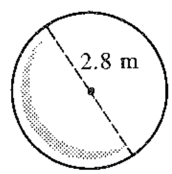
The surface area of the sculpture is approximately 32 m^2.

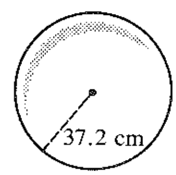
EXERCISES 12-7

(A)

1. Calculate the surface area and the volume of each sphere, to the nearest unit.

a) 

b) 

c) 

2. Find the surface area and the volume of each ball.

	Sport	Diameter of ball
a)	Basketball	25 cm
b)	Squash	13 mm
c)	Tennis	6.5 cm

(B)

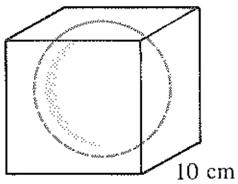
3. A balloon was blown up to the shape of a sphere, 20 cm in diameter. How much air did it contain?

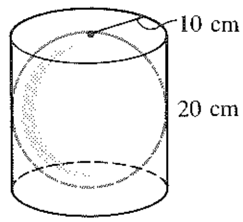
4. Find a formula for the volume of a sphere in terms of its diameter d.

5. The radii of the Earth, the moon, and the sun are shown. Calculate each surface area and volume.

		Radius
a)	Earth	6370 km
b)	Moon	1740 km
c)	Sun	694 000 km

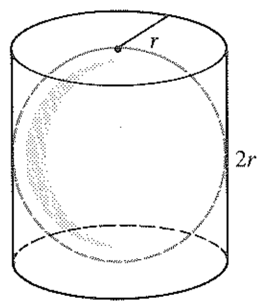
6. A sphere just fits inside a cube with edges of length 10.0 cm (below left). Calculate, to three significant digits:
   a) the surface area of the sphere b) the volume of the sphere.





7. A sphere just fits inside a cylinder with base radius 10.0 cm and height 20.0 cm (above right). Calculate, to three significant digits:
   a) the surface area of the sphere b) the volume of the sphere.

8. A sphere with radius r is contained in a cylinder with radius r and height $2r$, as shown.
   a) Determine the surface area of the sphere and the total surface area of the cylinder. How do the areas compare?
   b) Determine the total volume of the sphere and the total volume of the cylinder. How do the volumes compare?



9. A sphere has a surface area of 100 cm^2. Calculate the radius of the sphere, to the nearest tenth of a centimetre.

10. A sphere has a volume of 100 cm^3. Calculate the radius of the sphere, to the nearest tenth of a centimetre.

11. If we did not know that the moon's shape was spherical, we might think that it was a circle. How does the surface area of the part of the moon we see when the moon is full compare with the area of the circle that it appears to be?

(C)

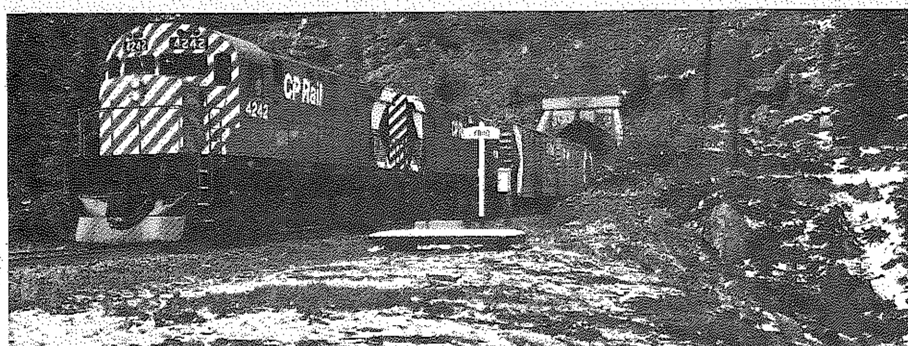
12. A balloon was blown up to the shape of a sphere with a circumference of 60 cm. Calculate, to three significant digits:
    a) the surface area of the balloon b) the volume of air it contained.

13. A spherical soap bubble has a radius of 2.0 cm. It lands on a flat surface and changes into a hemisphere.
    a) Assuming that none of the air inside the bubble escapes, calculate the radius of the hemisphere.
    b) What was the percent change in the surface area of the soap film?

14. The circumference of a sphere is C. Find expressions in terms of C for:
    a) the surface area b) the volume.

PROBLEM SOLVING

Choose the Strategy



1. A train 1 km long travels at 30 km/h through a tunnel 1 km long. How long does it take the train to clear the tunnel?

2. For which integers, n, do n and n^5 have the same ones digit?

3. a) Given 10 points on a circle, how many chords can be drawn joining them?
   b) How many chords can be drawn joining n points on a circle?

4. A sheet of 50 stamps is printed in 10 rows of 5 stamps. The edges of the stamps forming the sides of the sheet are straight. How many stamps have:
   a) 2 straight edges   b) 1 straight edge   c) no straight edge?

5. A girl is standing in line to buy a ticket. She observes that $\frac{1}{7}$ of all the people in line are in front of her while $\frac{5}{6}$ of all the people are behind her. How many people are in line?

6. a) Use the pattern suggested by the diagram to evaluate this expression.
   $1 + 2 + 3 + \ldots + 9 + 10 + 9 + \ldots + 3 + 2 + 1$
   b) Find a formula for this expression.
   $1 + 2 + 3 + \ldots + (n - 1) + n + (n - 1) + \ldots + 3 + 2 + 1$

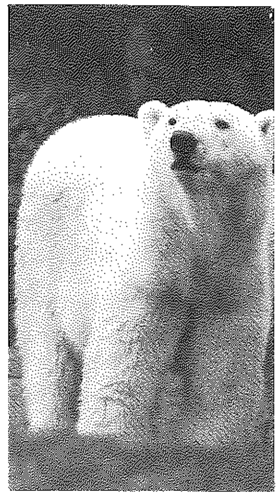
7. Estimate the probability that a hand of 4 cards dealt from a standard deck of 52 cards contains exactly 2 black cards and 2 red cards.

8. The cost of printing greeting cards is a fixed amount, plus a fixed rate per card. The total cost of printing 8 greeting cards is \$19.75 and the total cost of printing 20 greeting cards is \$34.75. What is the cost of printing 15 cards?

9. What is the least number of pieces into which a circular pie can be divided by n cuts, where a cut corresponds to a chord of the circular pie?

MATHEMATICS AROUND US

Why Polar Bears and Penguins Don't Freeze



Scientists studying different species of fox discovered that foxes living in warmer climates had long ears while those living in the cold climates had very short ears.

Further investigation revealed that animals living in the cold regions of the world have smaller appendages (limbs, ears, and tails) than animals in tropical areas. This principle is called *Allen's rule*.

Furthermore, it was discovered that polar animals tend to be bulkier; that is, less elongated than tropical animals. This principle is called *Bergmann's rule*.

Both Allen's and Bergmann's rules can be understood in mathematical terms. Animals produce heat in proportion to their body size but they lose heat in proportion to their surface area. Therefore, the animal which is shaped so that it has the smallest value of $\frac{\text{surface area}}{\text{volume}}$, is best adapted to a cold climate.

QUESTIONS

1. Consider the snake as a cylinder, with radius 0.5 cm and length 64 cm.

   Calculate the surface area and the volume of the snake, in terms of π. Write down the fraction, $\frac{\text{surface area}}{\text{volume}}$.

2. Consider the bear as a sphere, with radius 1.5 m.

   Calculate the surface area and the volume of the bear, in terms of π. Write down the fraction, $\frac{\text{surface area}}{\text{volume}}$.

3. Which of the two animals, the bear or the snake, has the better shape for surviving cold temperatures? Explain your answer.

4. If you started with a lump of plasticine, what shape would you mould it to have minimum surface area?

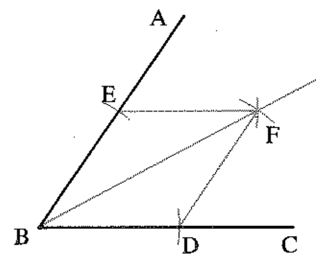
Review Exercises

1. a) Construct the bisector of an acute angle.
   b) Construct the perpendicular to a line through a point not on the line.
   c) Construct the perpendicular to a line through a point on the line.
   d) Construct the perpendicular bisector of a line segment.
   e) Construct an angle of 60°.

2. a) Construct $\triangle ABC$ with $BC = 8$ cm, $\angle B = 45°$, and $\angle C = 30°$.
   b) Construct the bisector of $\angle C$.
   c) Construct the median from B to AC.
   d) Construct the altitude from A to BC.

3. a) Construct $\triangle XYZ$ with $YZ = 6.5$ cm, $\angle Y = 75°$, and $\angle Z = 60°$.
   b) Construct the perpendicular bisector of XY.
   c) Construct the altitude from Y to XZ.

4. The diagram shows the construction for the bisector of $\angle ABC$ using ruler and compasses.

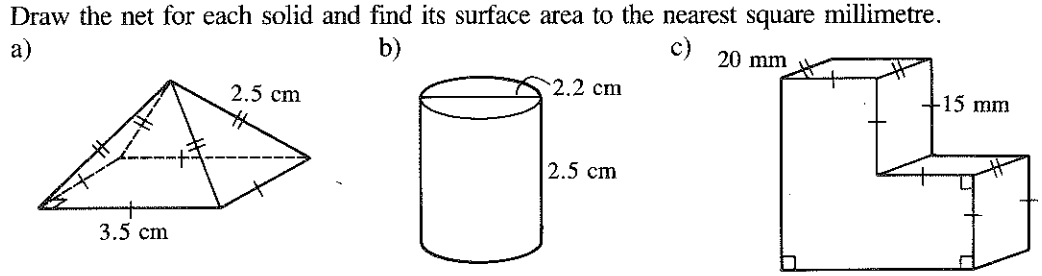


   a) Explain why:
      i) $FD = FE$
      ii) $BD = BE$
      iii) $\triangle FDB \cong \triangle FEB$.
   b) How does part a) explain why FB is the bisector of $\angle ABC$?

5. Construct a circle which passes through $A(-5,-6)$, $B(3,-4)$, and $C(1,4)$.

6. Draw the net for each solid.
   a) a 2 cm cube
   b) a tetrahedron with 3 cm edges
   c) a 12 cm high prism having a square base with 1.5 cm edges
   d) a 5 cm high cylinder with the radius of the base 2 cm

7. Draw the net for each solid and find its surface area to the nearest square millimetre.

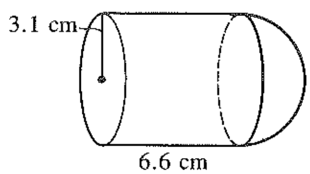


8. The Far North Pavilion at Ontario Place, Toronto, consists of a number of cylindrical silos of various sizes. One silo, with a diameter of 10.2 m, is 8.6 m high. It is to be painted with paint costing \$3.75/L. If 1 L of paint covers 5.3 m², what will be the total cost of the paint?

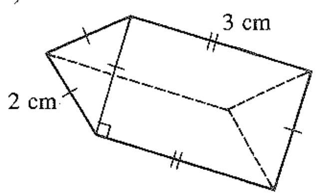
9. A pyramid, 12 m high, has a square base with 15 m edges. How much would it cost to paint the pyramid if 1 L of paint costs \$5.50 and covers 7.2 m²?

10. Calculate each volume to the nearest unit.

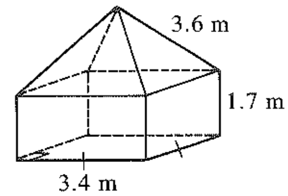
a)



b)



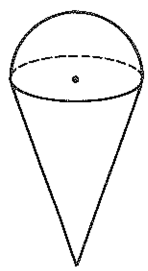
c)



11. Tennis balls are packed in cans 8.4 cm in diameter and 25.5 cm high. Three dozen of these cans are packed in a box 51 cm by 51 cm by 26 cm. What is the total volume of wasted space?

12. An ice-cream manufacturer plans to build a giant ice-cream cone as an advertisement for its product. The cone will be 2 m high and 1 m in diameter. The ice cream in the top of the cone will be hemispherical in shape.

Four litres of paint cover 35 m^2. How much paint will be needed for 3 coats?



13. A pyramid is supposed to have magical properties. A manufacturer built a pyramid large enough for a person to sit inside and be rejuvenated! The pyramid is 1 m high and its square base has edges of length 1 m. What is the volume of air inside the pyramid, to the nearest tenth of a cubic metre?

14. A smaller pyramid is built, which the manufacturers claim will sharpen a razor blade that is hung inside. The pyramid is 12 cm high and has a square base of edge length 8 cm. The pyramid does not have a base. What is the area of the material necessary to build the pyramid, to the nearest square centimetre?

15. A paper cup has a conical shape, with a base diameter of 5 cm and a height of 7 cm.
   a) Find the volume of water a cup will contain when it's filled to the brim.
   b) How high, to the nearest millimetre, would the level of water be if it were poured into a cylindrical tumbler with the same radius?

16. A cylindrical can of orange concentrate has a diameter of 6 cm and a height of 12 cm. The instructions on the can are to mix the orange with 3 cans of water.
   a) Will the mixture fit into a cylindrical jug with diameter 10 cm and height 20 cm?
   b) How high would a jug with diameter 10 cm have to be, to hold the mixture?
   c) How many paper cups from *Exercise 15* would the mixture fill?

17. An apartment building in Aylmer, Ontario, is heated by solar heating. Water is stored in a buried cylindrical tank 6.7 m deep and 15 m in diameter.
   a) What area of material was needed to make the tank, to the nearest square metre?
   b) What is the volume of the tank, to the nearest cubic metre?

Cumulative Review, Chapters 10-12

1. Triangle ABC has vertices A(−1,4), B(−1,1), and C(4,1). Draw the image of △ABC under:
   a) a translation which maps P(3,5) onto P′(6,7)
   b) a reflection in the line $y = -1$
   c) a rotation of 90° about the origin
   d) a dilatation with a scale factor of 2 and rotation centre the origin.

2. A translation maps (−2,1) onto (3,−1).
   a) Find the images of P(2,4), Q(−1,0), and R(4,−2) under this translation.
   b) If D′(−4,2), E′(−2,−2), and F′(1,3) are image points under this translation, find the coordinates of the original points.

3. The reflection images of A(3,8), B(1,3), and C(6,7) are A′(8,3), B′(3,1), and C′(7,6). Graph these points and draw the reflection line.

4. The dilatation image of a figure has an area of 450 cm². Find the area of the original figure for each dilatation factor.

   a) 3 b) $\frac{5}{8}$

5. The cenotaph in the town square casts a shadow of length 12.8 m. At the same time, a 1.5 m post casts a 2.1 m shadow. How tall is the cenotaph?

6. A 2.4 kg sample of nuts is found to contain 384 g of pecans, 864 g of peanuts, 768 g of hazel nuts, and the rest are cashews.
   a) What is the mass of the cashews?
   b) Draw a circle graph to display this information.
   c) What percent of the nuts are:
      i) pecans ii) peanuts iii) cashews?

7. The maximum daily temperatures in Vancouver, during July, over a five-year period are shown in this table.

Maximum Temperature (°C)	13-15	16-18	19-21	22-24	25-27	28-30	31-33
Frequency (days)	4	14	22	32	40	30	13

   Draw a histogram for this data.

8. Find the measures of central tendency for the following mathematics marks.

82	78	75	72	68	88	98	67	65	75
43	53	69	86	64	81	72	34	80	71
67	72	41	75	75	73	54	74	73	70

9. If each number in a set of data is increased by 3, how would:
   a) the mean b) the median and c) the mode change?

10. The probability of getting 4 heads on the toss of 4 coins is $\frac{1}{16}$.
    a) What is the probability of not getting 4 heads?
    b) What is the probability of getting 4 tails?

11. A pair of dice is rolled. Let A denote the event that the total number shown is even. Let B denote the event that the total number shown is less than 7.
 a) What outcomes are favorable to:
  i) event A ii) event B iii) both events A and B?
 b) What is the probability of:
  i) event A ii) event B iii) events A and B together?

12. In a card game known as "In Between", a player is dealt two cards. To win, the player must be dealt a third card with a value in between the first two. Calculate the probability of winning if the two cards already dealt are:
 a) a three and a seven b) a nine and a Queen
 c) a four and a ten d) a six and a seven

13. What is the probability that in any year chosen at random the month of February has:
 a) exactly 28 days b) 29 days
 c) 30 days d) more than 27 days?

14. Kara says that the probability that a person can cross a street safely is $\frac{1}{2}$ because there are two possible outcomes, crossing safely or not crossing safely. Do you agree? Why?

15. Construct $\triangle$PQR with PQ perpendicular to QR, R = 60°, and QR = 3.5 cm. Construct the median from P to QR and the bisector of $\angle$Q.

16. Construct a circle which passes through the points A(−2,1), B(8,4), and C(5,−3).

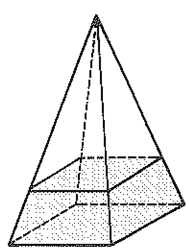
17. Find the surface area of:
 a) a cube of side 5.6 cm
 b) a cylinder 16 cm long and 3 cm in diameter c) a sphere of radius 7.5 cm.

18. Find the volume of:
 a) a rectangular prism 28 cm long, 12 cm wide, and 8 cm high
 b) a cone of radius 10 cm and height 32 cm
 c) a cylinder of radius 6.4 cm and height 22 cm
 d) a pyramid with a 15 cm square base and a height of 40 cm.

19. How much oil is in a barrel of height 1.2 m and radius 0.65 m?

20. What would it cost to paint the outside of 1200 barrels similar to the one in *Exercise 19* if the paint costs $8.40/L and 1 L covers 9.5 m^2?

21. A square pyramid with the length of a side of the base 24 cm and height 52 cm is cut. A pyramid 39 cm high is formed. What is the volume of the remaining portion?



Answers

The Nature of Mathematics

The Search for Pattern

Exercises, page xiii

1. a) $1 + 3 + 5 + 7 = 4 \times 4$
$1 + 3 + 5 + 7 + 9 = 5 \times 5$
$1 + 3 + 5 + 7 + 9 + 11 = 6 \times 6$
b) $2 + 4 + 6 + 8 = 4 \times 5$
$2 + 4 + 6 + 8 + 10 = 5 \times 6$
$2 + 4 + 6 + 8 + 10 + 12 = 6 \times 7$

2. a) 45, 56, 67 **b)** 23, 28, 32 **c)** 81, 243, 729
d) 2, 1, 0.5 **e)** 416, 421, 429
f) 1847, 1942, 2037

3. a) 17, 20, 23 **b)** Row 1

4. a) 32, 64, 128 **b)** 16, 22, 29 **c)** 11, 13, 14

5. Answers may vary, for example,
1, 2, 3, 4, 5, 6, ...; 1, 2, 3, 1, 2, 3, 1, 2, 3, ...;
1, 2, 3, 5, 7, 10, 13, ...

6. 10, 15, 21 **7.** 16, 25, 36

8. a) 7 **b)** 3 **c)** 1 **d)** 3

The Value of Mathematical Investigation

Exercises, page xv

1. a) 14 **b)** 24 **c)** 36

2. b) i) 14 **ii)** 24 **iii)** 36 **3.** 35

The Power of Mathematical Reasoning

Exercises, page xvii

1. a) Yes **b)** No **c)** No **d)** No

3. a), c)

The Power of the Calculator

Exercises, page xix

1. a) 2.283 333 3 **b)** 3.103 210 7
c) 7.071 031 7

2. a) 63, 6633, 666 333;
$9999 \times 6667 = 66\ 663\ 333$
b) 24, 2244, 222 444;
$6666 \times 3334 = 22\ 224\ 444$
c) 81, 9801, 998 001;
$9999 \times 9999 = 99\ 980\ 001$
d) 25, 4225, 442 225;
$6665 \times 6665 = 44\ 422\ 225$
e) 10 201, 40 804, 91 809;
$404 \times 404 = 163\ 216$
f) 11, 111, 1111; $1234 \times 9 + 5 = 11\ 111$

3. Answers may vary.

4. Yes, because 3 consecutive numbers contain the factors 2 and 3.

5. a) 1 048 576 **b)** Answers may vary, with an 8-digit display, 2^{26} **c)** 2^{29}

6. a) 40 320 **b)** 11 **c)** 14

7. Scientific notation 1×10^8

8. a) 2.4×10^{10} **b)** 3.7×10^{12} **c)** 2.913×10^9

9. a) 9.83 08 **b)** 1.03 10 **c)** 2.345 09

The Power of the Computer

Exercises, page xxi

1. a) 3.597 739 66 **b)** 4.499 205 34
c) 5.878 030 95 **d)** 6.792 823 42
e) 0.447 247 47 **f)** 0.914 792 47 **2.** 900

3. a) 31 **b)** 83 **c)** 227 **d)** 616 **e)** 1674

The Utility of Mathematics

Exercises, page xxiii

1. a) About 27.82 km **b)** About 464 m

2., 3. Answers may vary.

4. a) i) About 940 000 000 km
ii) About 2 600 000 km **b)** About 110 000 km/h

5. a) About 1800 km **b)** About 30 km

A Famous Unsolved Problem

Exercises, page xxiv

1. 6:3,5; 5,7; 11,13; 17,19; 29,31; 41,43

2. a) 59, 61; 71, 73 **b)** 101,103; 107,109

3. Answers may vary. **a)** 3,7 **b)** 5,13 **c)** 11,19
d) 13,23 **e)** 17,29 **f)** 17,31 **g)** 3,19

4. No, because there might be some after 100 000 000.

5. 2,5; because one prime would have to be even, and 2 is the only prime that is even.

6. Because there is only one even prime number.

7. One number is odd, one is even, hence their product is even. **8.** 17

Chapter 1

Exercises 1-1, page 4

1. B, −7; C, −5; F,0; H,4; I,6; K,9

2. a) +\$9 **b)** −\$21 **c)** +80°C **d)** −20°C
e) +\$50 **f)** −\$75 **g)** −\$81
h) −12 000 m **i)** +3000 m

3. a) a loss in altitude of 300 m
b) a gain in altitude of 25 m
c) a loss in altitude of 100 m
d) a gain in altitude of 2 m

4. a) a debt of \$12 **b)** a credit of \$7
c) a credit of \$15 **d)** a debt of \$53