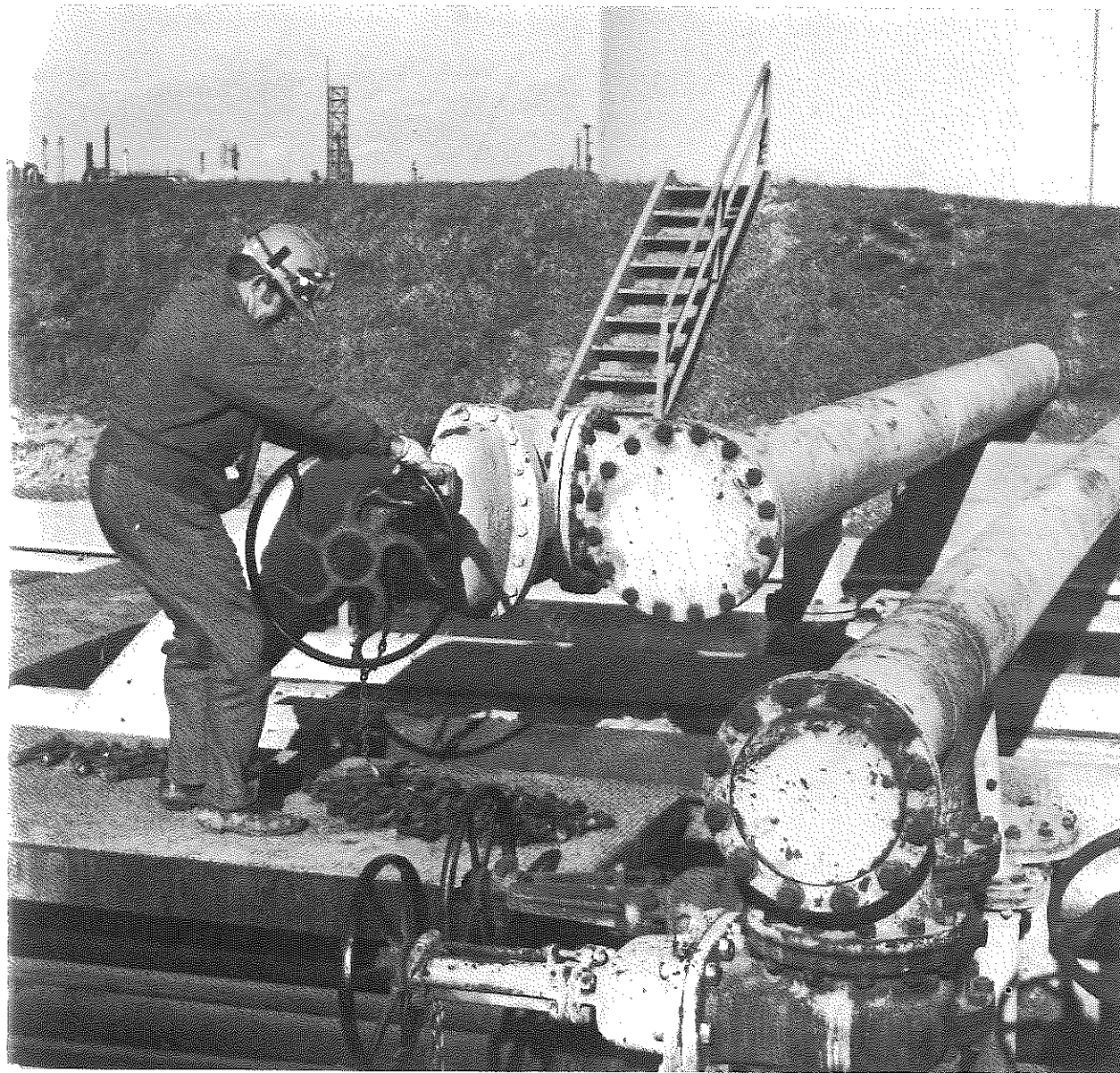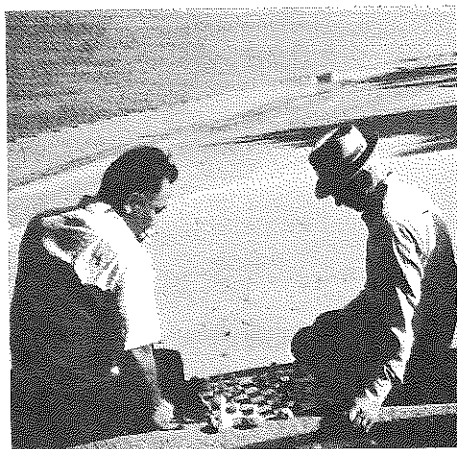# 10 Transformations

Two oil storage tanks, several kilometres apart, are on the same side of a pipeline. Where, along the pipeline, should a pumping station be located to serve both tanks so that the total length of pipe from the pipeline to the tanks is a minimum? (See Section 10-7, *Example 1*.)

10-1 INTRODUCTION TO TRANSFORMATIONS

The photographs above show some familiar transformations.

Whenever the shape, the size, the appearance or the position of an object is changed, it has undergone a *transformation*. Under a transformation, some of the characteristics of an object may be changed while others remain the same. Those characteristics that are unchanged are said to be *invariant*.

For each photograph above, identify some characteristics of the transformation that are invariant as well as some that change.

Transformation geometry is the study of transformations of geometric figures. Three transformations that do not change size or shape are *rotations, reflections,* and *translations.* These transformations map a figure onto its image.
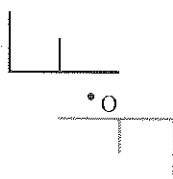
A translation, whose direction and length are illustrated by the colored translation arrow, maps the black figure onto its colored image.
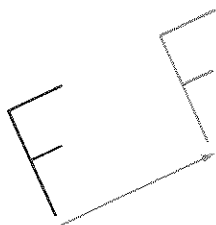
A reflection in the line *l* maps the black figure onto its colored image.
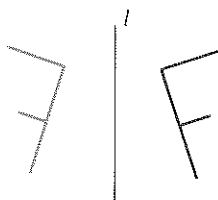
A rotation of 180° about O maps the black figure onto its colored image.

Example 1. State which transformation maps each black letter onto its colored image.

a)   b)   c)

Solution.   a) Since the black G maps onto the colored G by a rotation about a point between the letters, the diagram illustrates a rotation.

            b) Since the black G maps onto the colored G by a translation along a line joining the letters, the diagram illustrates a translation.

            c) Since the black G maps onto the colored G by a reflection in a line between the letters, the diagram illustrates a reflection.
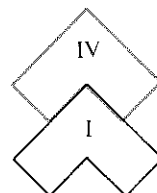
Example 2. The four parts into which the
figure is divided are congruent.
Name the transformation that
maps:

a) region I onto region IV

b) region II onto region IV

c) region I onto region III
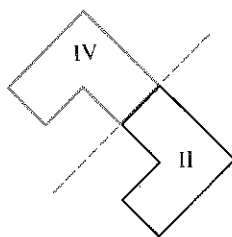
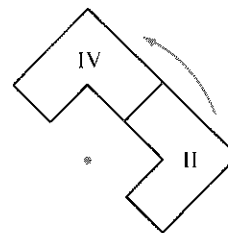d) region III onto region IV.
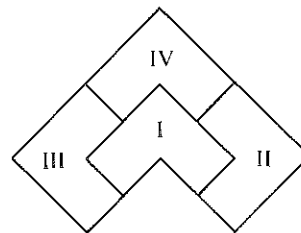
Solution. a) Region IV is a translation
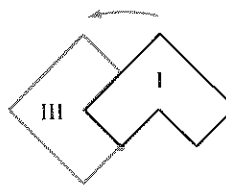image of region I.
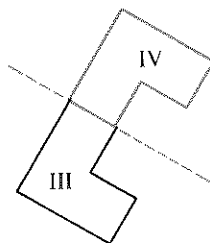
b) Region IV is a reflection
image of region II.

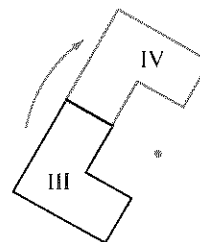Region IV is also a rotation
image of region II.

c) Region III is a rotation
image of region I.

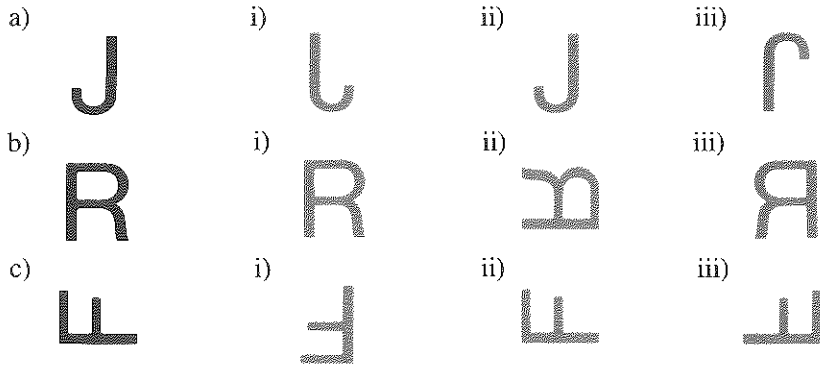d) Region IV is a reflection
image of region III.

Region IV is also a rotation
image of region III.

EXERCISES 10-1

Ⓐ

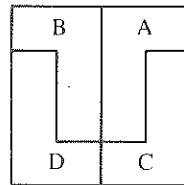1. Name the transformation required to map each letter onto its colored image.

a)
J
i) L
ii) J
iii) ∩

b)
R
i) R
ii) Я
iii) Я

c)
F
i) F
ii) F
iii) F

2. For the figure shown, name a transfor-
   mation that maps:
   a) region 4 onto region 1
   b) region 3 onto region 4
   c) region 4 onto region 2
   d) region 1 onto region 2
   e) region 3 onto region 1
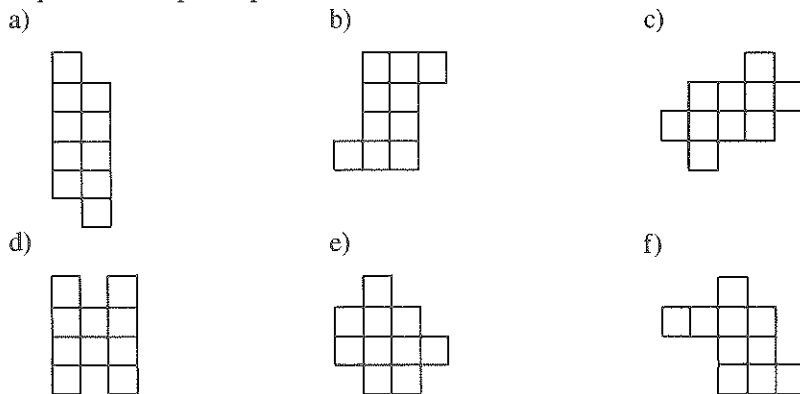   f) region 2 onto region 3.
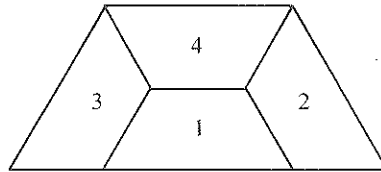
3. For the figure shown, name a transfor-
   mation that maps:
   a) region C onto region A
   b) region A onto region B
   c) region D onto region C.

Ⓑ

4. Copy each diagram. Divide it into two congruent parts. Name the transformation
   required to map one part onto the other.

a)

b)

c)

d)

e)

f)

INVESTIGATE

1. Plot the points V(-3,6), W(1,1), and Z(-1,-8), and draw △VWZ.

2. Translate △VWZ as follows: move every point 7 units in the negative *x*-direction and 3 units in the positive *y*-direction.

3. Label the image points V', W', and Z'. Then, △V'W'Z' is the translation image of △VWZ.
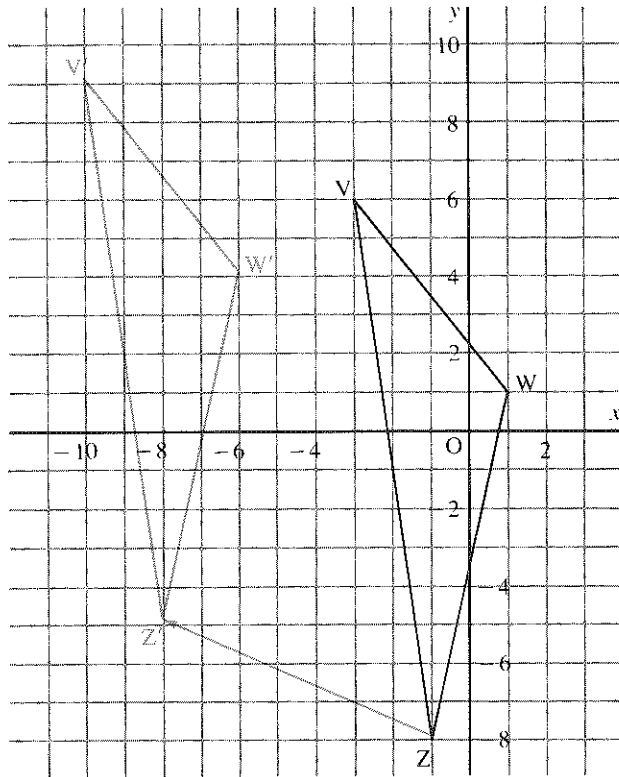
4. Measure VW, WZ, and VZ.

5. Measure V'W', W'Z', and V'Z'.
   - What appears to be true about the lengths of line segments under a translation?

6. Measure ∠VWZ, ∠WZV, and ∠ZVW.

7. Measure ∠V'W'Z', ∠W'Z'V', and ∠Z'V'W'.
   - What appears to be true about the measures of angles under a translation?
   - What appears to be true about the direction of line segments under a translation?
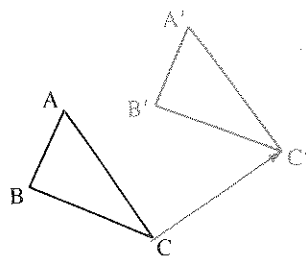
10-2 TRANSLATIONS

When a hockey puck moves across the ice in a straight line, it is *translated* from one position to another.

When a point or a figure is moved (or translated) in a straight line to another position in the same plane, it is said to have undergone a *translation*.

The translation illustrated here is defined by the length and the direction of the colored arrow.

$\triangle A'B'C'$ is the translation image of $\triangle ABC$.
$AB = A'B'$; $AC = A'C'$; $BC = B'C'$
$AB \parallel A'B'$; $AC \parallel A'C'$; $BC \parallel B'C'$
$\angle ABC = \angle A'B'C'$; $\angle BAC = \angle B'A'C'$;
$\angle ACB = \angle A'C'B'$

Under a translation, any figure and its image are identical in all respects, except location.

You may have discovered the following properties of a translation in the previous *INVESTIGATE*.


• Under a translation, the lengths of line segments are invariant.
• Under a translation, the directions of line segments are invariant; that is, the image line segment is parallel to the original line segment.
• Under a translation, the measures of angles are invariant.

We use these properties to solve problems involving translations.

**Example 1.** Describe the translation that maps F onto F'.
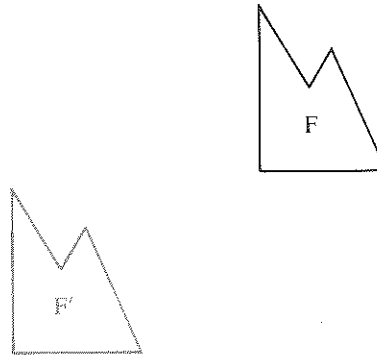
Solution. Trace the diagram. Join corresponding points Z and Z' with a straight line segment.
Draw an arrow head at Z', pointing to Z'.
The length and direction of the arrow ZZ' represent the translation that maps F onto F'.
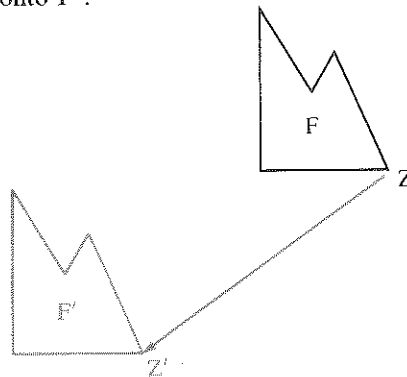
Example 2. A triangle has vertices G($-1,3$), H($4,7$), and J($3, -3$).
  a) Draw the image of △GHJ under this translation: move every point 5 units in the positive x-direction and 4 units in the negative y-direction.
  b) Write the coordinates of the vertices of the image triangle.

Solution. a) The translation means that every point on △GHJ slides 5 units to the right and 4 units down. Thus, the x-coordinate of each point increases by 5 and the y-coordinate decreases by 4.
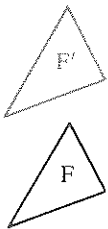△G'H'J' is the image of △GHJ under the given translation.
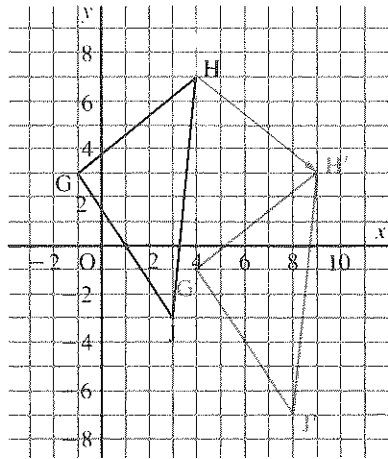
b) $G(-1,3) \rightarrow G'(4,-1)$
$H(4,7) \rightarrow H'(9,3)$
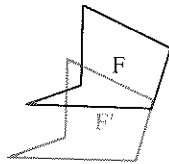$J(3,-3) \rightarrow J'(8,-7)$

EXERCISES 10-2

Ⓐ

1. Each diagram shows a figure F and its image F' after a translation. Trace each diagram. Draw the translation arrow.
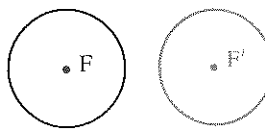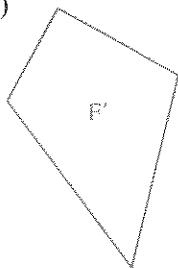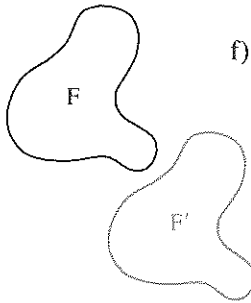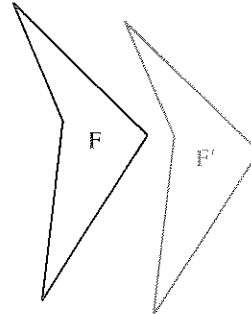
a)

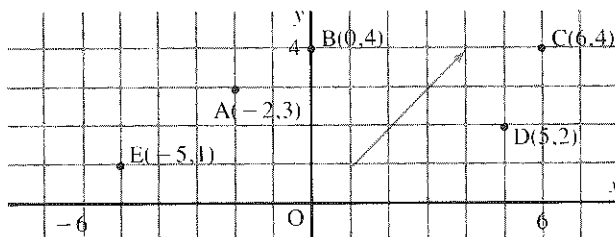

b)

c)

d)

e)

f)

2. A translation maps the point $(1, -2)$ onto $(3,0)$.
   a) Plot the points and draw the translation arrow.
   b) Find the images of these points under this translation.
      i)   A(0,2)   ii) B(5,6)   iii) C(-3,-1)   iv) D(-4,2)

3. The graph shows points A, B, C, D, E, and a translation arrow.

   a) Plot the points and their images under the translation. Label the images A', B', C', D', and E'.
   b) Draw line segments AA' and DD', and compare them with the translation arrow. What do you notice?
   c) Measure and compare the lengths of line segments DE and D'E'. What do you notice?
   d) Measure and compare the sizes of $\angle$AED and $\angle$A'E'D'. What do you notice?

4. A translation maps the point $(-2, -3)$ onto $(4,2)$.
   a) Plot the points and draw the translation arrow.
   b) Find the images of A(-1,1), B(1,4), and C(2,-3) under this translation. Label the image points A', B', and C'.
   c) Describe the effect of this translation on $\triangle$ABC.

5. A parallelogram has vertices at A(-2,2), B(2,1), C(4,-4), and D(0,-3). A translation maps points 4 units to the right and 1 unit down. Draw:
   a) the parallelogram
   b) the translation arrow
   c) the image of the parallelogram under this translation.

6. A translation maps the point $(2,5)$ onto $(5, -2)$.
   a) Draw the translation arrow.
   b) If P'(-3,-1), Q'(-1,3), and R'(-5,0) are the images of P, Q, and R under this translation, find the coordinates of P, Q, and R.

7. What properties remain invariant under a translation?

8. a) Graph this equation. $2x + y = 6$
   b) Draw the image of the graph of the line in part a) under the translation that maps the point $(3,0)$ onto $(0,0)$.

9. a) Graph this equation. $5x - 2y = 10$
   b) Draw the image of the graph of the line in part a) under the translation that maps the point $(-2,1)$ onto $(0,6)$.
   c) Explain the result.

10. Translation T_1 maps the point $(4,1)$ onto $(2,3)$. Translation T_2 maps $(-2,-3)$ onto $(3,0)$.
   a) Draw $\triangle ABC$ and its image, $\triangle A'B'C'$, under T_1.
   b) Draw the image of $\triangle A'B'C'$ under T_2. Label it $\triangle A''B''C''$.
   c) Draw a translation arrow for the single translation that maps $\triangle ABC$ onto $\triangle A''B''C''$.
   d) Investigate whether T_1 followed by T_2 gives the same result as T_2 followed by T_1.
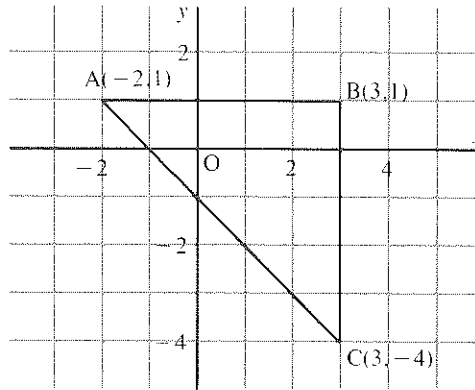
INVESTIGATE

1. Plot the points $S(-3,-5)$, $T(-1,7)$, and $U(-1,-8)$, and draw $\triangle STU$.

2. Reflect $\triangle STU$ in the y-axis. That is, fold the paper along the y-axis. With a sharp pencil or compasses point, mark the positions of S, T, and U so they appear on the other half of the folded paper.
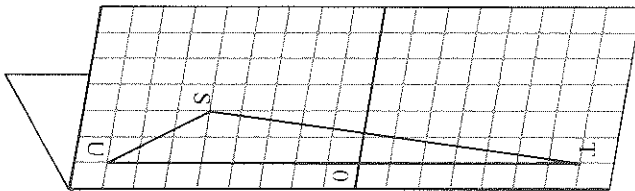
3. Open the paper and label the image points S', T', and U'. Then, $\triangle S'T'U'$ is the image of $\triangle STU$ under a reflection in the y-axis.

4. Measure ST, TU, and SU.

5. Measure $S'T'$, $T'U'$, and $S'U'$.
   ● What appears to be true about the lengths of line segments under a reflection?

6. Measure $\angle STU$, $\angle TUS$, and $\angle UST$.

7. Measure $\angle S'T'U'$, $\angle T'U'S'$, and $\angle U'S'T'$.
   ● What appears to be true about the measures of angles under a reflection?
   ● Where is the image of any point on the reflection line?
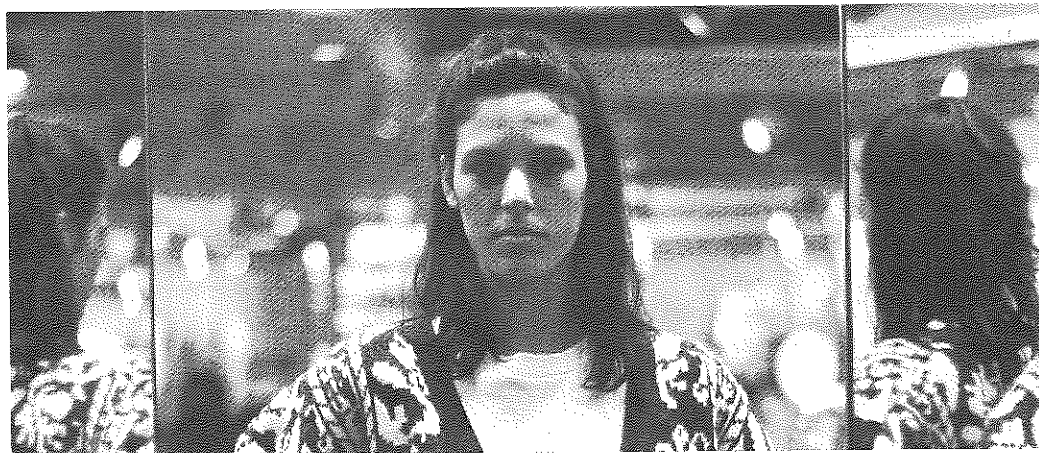
10-3 REFLECTIONS

When you look in a mirror, you see an image of yourself. The image appears to be as far behind the mirror as you are in front of it. The transformation that relates points and their images in this way is called a *reflection*.

A reflection in line l maps each point on a figure, onto its image point on the reflection of the figure.

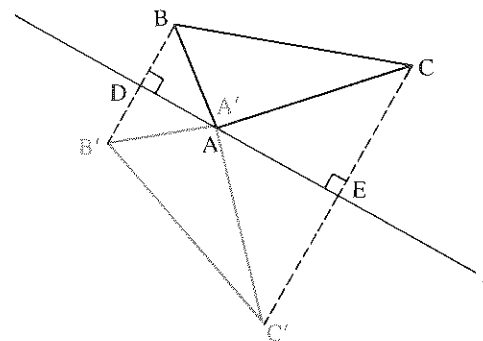$\triangle A'B'C'$ is the reflection image of $\triangle ABC$ in reflection line l.

$AB = A'B'; AC = A'C'; BC = B'C'$

$\angle ABC = \angle A'B'C'; \angle BAC = \angle B'A'C';$

$\angle ACB = \angle A'C'B'$

The points A and A' coincide; that is, a point on the reflection line is its own image.

In the diagram, each point on $\triangle ABC$ has been joined to its image point on $\triangle A'B'C'$, with a broken line.

$BD = DB'; CE = EC'$

$\angle BDA = \angle B'DA = 90°; \angle CEA = \angle C'EA = 90°$

You may have discovered the following properties of a reflection in the previous *INVESTIGATE*.

* Under a reflection, the lengths of line segments and the measures of angles are invariant.
* Under a reflection, the points on the reflection line are invariant.
* The reflection line is the perpendicular bisector of the line segment joining any point to its image point.

We use these properties to solve problems involving reflections.

Example 1. Draw the reflection line that maps F onto F'.
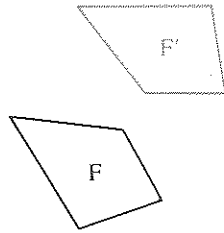
Solution. Trace the diagram.
Join corresponding points X and X'
with a straight line segment and mark
its midpoint M.
Join another pair of corresponding
points Y and Y' with a straight line
segment and mark its midpoint N.
Join MN. This is the reflection
line that maps F onto F'.
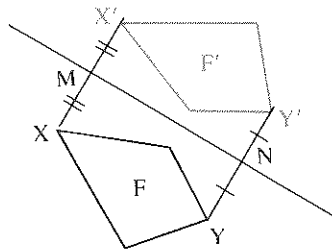
Example 2. A triangle has vertices D(4,7), E(8,3), and F(6, −2). Draw the image of
△DEF after a reflection in the *x*-axis. Write the coordinates of the
vertices of the image triangle.

Solution. Plot the points on graph paper. Draw △DEF.
Draw a line through D perpendicular to the *x*-axis (that is, parallel
to the *y*-axis). Extend this line to a point D' such that D' is the same
distance from the *x*-axis as D is.

Similarly, draw a line
through E perpendicular to the
x-axis and label E' such that
E' is the same distance from
the *x*-axis as E is.

Also, draw a line through F
perpendicular to the *x*-axis
and label F' such that F' is the
same distance from the *x*-axis
as F is.

△D'E'F' is the image of
△DEF under a reflection in the
x-axis. The coordinates of the
vertices of the image triangle
are D'(4, −7), E'(8, −3), and
F'(6,2).

EXERCISES 10-3

Ⓐ
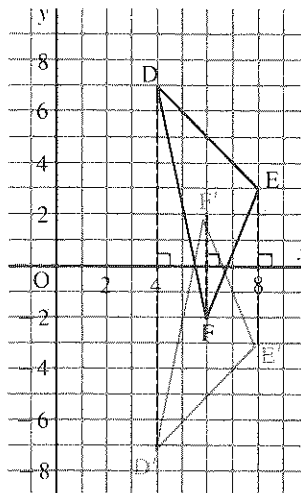
1. Sketch the images of **A**, **3**, **K**, and **5** under reflections in:
   a) a vertical line   b) a horizontal line.

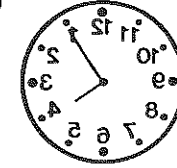2. State the time shown on the mirror image of each clock face.
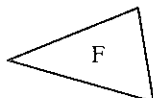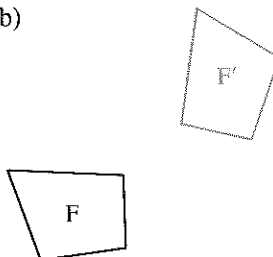   a)   b)   c)   d)
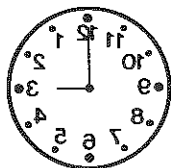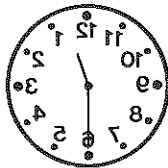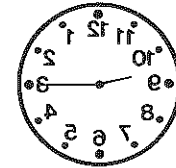
3. Each diagram shows a figure F and its image F′ under a reflection. Trace each diagram and draw the reflection line.
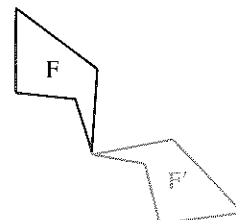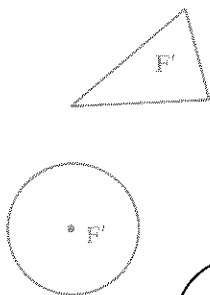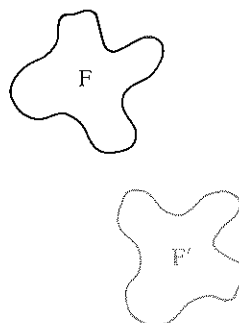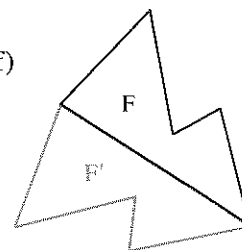   a)   b)   c)

   d)   e)   f)

Ⓑ

4. Triangle PQR has vertices P(2,6), Q(6,4), and R(3,2). Draw the image of △PQR under a reflection in:
   a) the *x*-axis   b) the *y*-axis.

5. Quadrilateral ABCD has vertices A(2,5), B(6,5), C(9,1), and D(2,1). Find the coordinates of the image of quadrilateral ABCD under a reflection in:
   a) the *x*-axis   b) the *y*-axis.

6. Triangle ABC has vertices A(4,7), B(7,2), and C(3,3). Its image under a reflection is A′($-2,7$), B′($-5,2$), and C′($-1,3$). Graph both triangles and draw the reflection line.

7. Triangle ABC in *Exercise 6* has another image at A'(4, −11), B'(7, −6), C'(3, −7). Draw the reflection line.
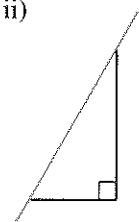
8. For each triangle shown below
   a) Copy the triangle.
   b) Draw its reflection image using the extended side as the reflection line.
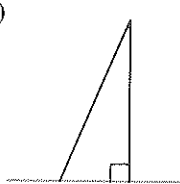   c) Name the figure formed by the triangle and its image.
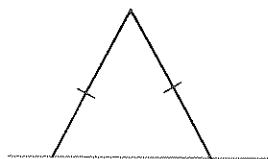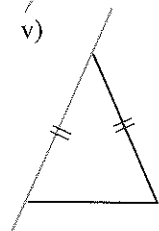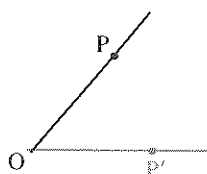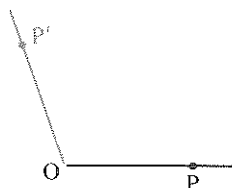   i) ii) iii)
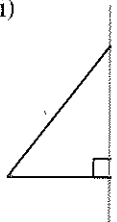
   iv) v) vi)

9. Each diagram shows a line OP and its reflection image OP'.
   a) Trace each diagram.
   b) Determine its reflection line.
   c) Verify that the reflection line is the bisector of ∠POP'.
   i) ii) iii)
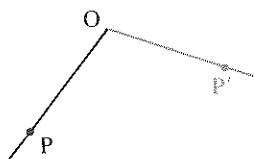
10. Triangle PQR has vertices P(−2,4), Q(4,2), and R(1, −2). Draw the image of △PQR and write the coordinates of the vertices under a reflection in:
    a) the x-axis   b) the y-axis.

11. Quadrilateral ABCD has vertices A(−2,6), B(4,3), C(3, −3), and D(−5, −2). Write the coordinates of the vertices of its image under a reflection in:
    a) the x-axis   b) the y-axis.

12. What properties remain invariant under a reflection?

13. a) Graph this equation. $3x + 2y = 12$
    b) Draw the image of the graph of the line in part a) under a reflection in:
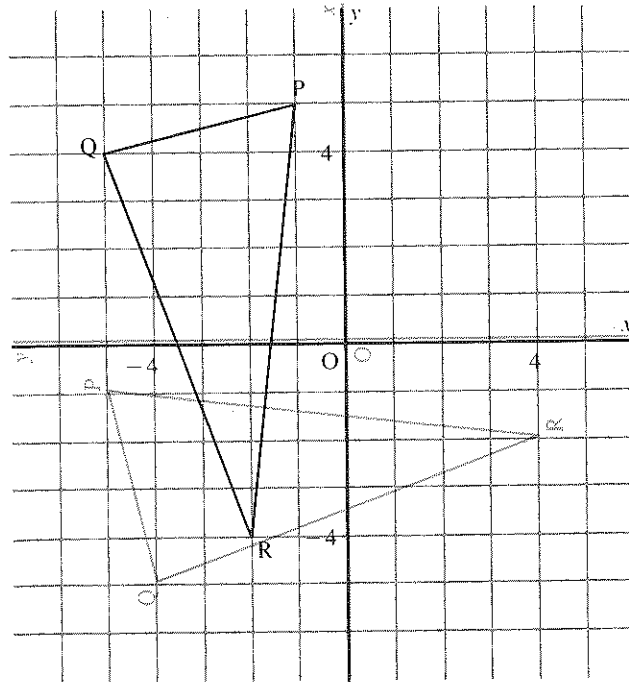   i) the x-axis   ii) the y-axis.

Rotations on a Grid

1. Plot the points P($-1,5$), Q($-5,4$), and R($-2,-4$), and draw $\triangle$PQR.

2. Trace the triangle and the axes on tracing paper. Rotate the tracing paper through 90° counterclockwise about the origin. That is, rotate the paper until the positive *y*-axis on the tracing paper coincides with the negative *x*-axis on the base paper.


3. Mark the new positions of P, Q, and R and label them P′, Q′, and R′. $\triangle$P′Q′R′ is the rotation image of $\triangle$PQR after a counterclockwise rotation of 90° about O.

4. Measure PQ, QR, and PR.

5. Measure P′Q′, Q′R′, and P′R′.
   * What appears to be true about the lengths of line segments under a rotation?

6. Measure $\angle$PQR, $\angle$PRQ, and $\angle$RPQ.

7. Measure $\angle$P′Q′R′, $\angle$P′R′Q′, and $\angle$R′P′Q′.
   * What appears to be true about the measures of angles under a rotation?

10-4 ROTATIONS

A person on a Ferris wheel rotates about the centre of the wheel. When a point or a figure is turned about a fixed point, it is said to have undergone a *rotation*. The fixed point is called the *rotation centre*. A rotation in a counterclockwise direction is a positive rotation. In this chapter, all rotations will be counterclockwise.

A counterclockwise rotation of 90° about O maps each point on a figure, onto its image point on the rotation image.
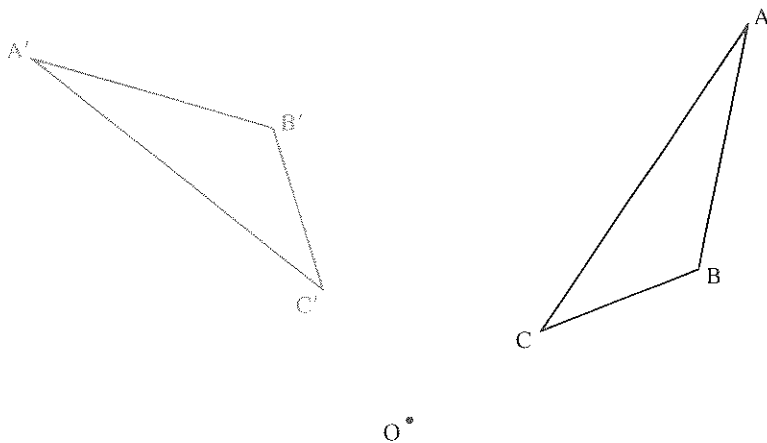
$\triangle A'B'C'$ is the rotation image of $\triangle ABC$ after a 90° rotation about O.
AB = A'B'; AC = A'C'; BC = B'C'
$\angle ABC = \angle A'B'C'$; $\angle BAC = \angle B'A'C'$; $\angle ACB = \angle A'C'B'$

You may have discovered the following properties of a rotation in the previous *INVESTIGATE*.

* Under a rotation, the lengths of line segments are invariant.
* Under a rotation, the measures of angles are invariant.

We can use these properties to solve problems involving rotations.

Example 1. Figure F′ is the image of figure F under a rotation.
  a) Locate the rotation centre.
  b) Measure the rotation angle.
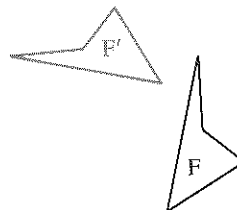
Solution. a) Trace the diagram on tracing paper. Fold the paper so that a point X on the figure coincides with the corresponding point X′ on the image. Crease the paper.
Unfold the paper and draw a line *l* along the crease.
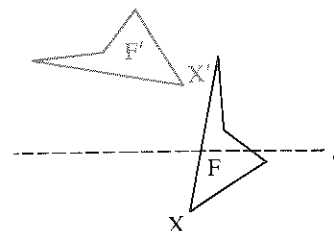
Fold the paper so another point Y on the figure coincides with its corresponding point Y′ on the image. Crease the paper, then unfold it and draw a line *m* along the crease.

The point of intersection O of *l* and *m* is the centre of rotation that maps the figure onto its image.
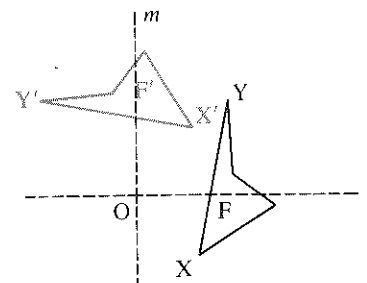
  b) Draw lines from O to X and from O to X′.

  ∠XOX′ is the rotation angle and it is measured to be 90°.

How could you check that the rotation centre and the rotation angle are correct?
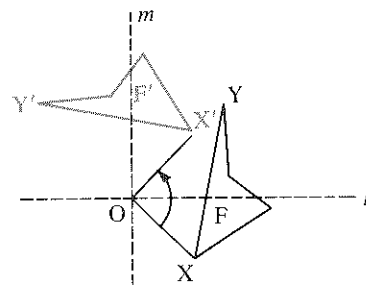
Example 2. A triangle has vertices A(3,5), B(7,1), and C(5, − 2).

a) Draw the image of △ABC under a rotation of 90° about the origin.

b) Write the coordinates of the vertices of the image triangle.

Solution. a) Plot the points on graph paper. Draw △ABC.
Draw OA. Place a protractor along OA, centre O and mark
90° counterclockwise. Along this line, label a point A′ such that
OA = OA′.
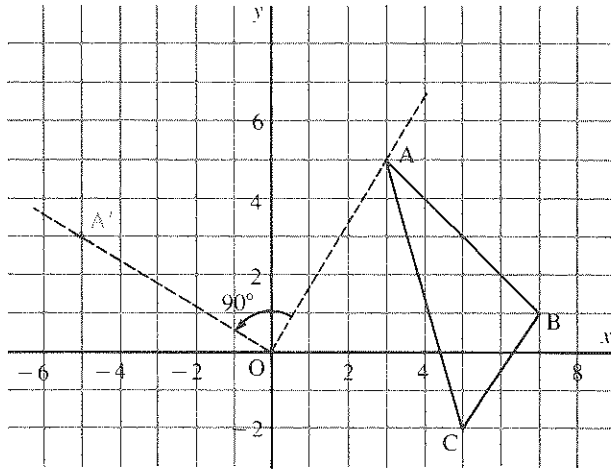
Similarly, mark B′ such that ∠BOB′ is 90° and OB = OB′.
Then mark C′ such that ∠COC′ is 90° and OC = OC′.
△A′B′C′ is the image of △ABC under a rotation of 90° about O.
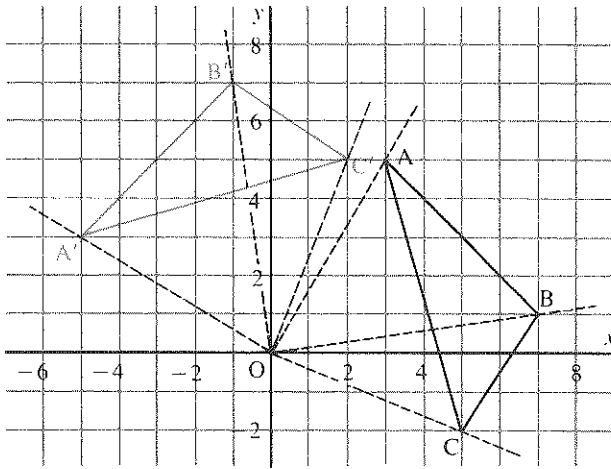
b) The coordinates of the vertices of the image triangle are A′(− 5,3),
B′(− 1,7), and C′(2,5).

EXERCISES 10-4

Ⓐ

1. Each diagram shows a figure F and its image F′ after a rotation. Trace each diagram. Locate the rotation centre and measure the rotation angle.

a)

b)

c)

d)

e)

f)

Ⓑ
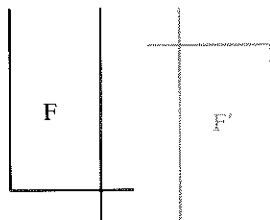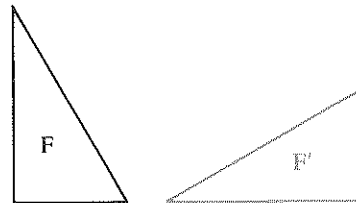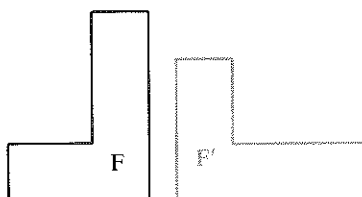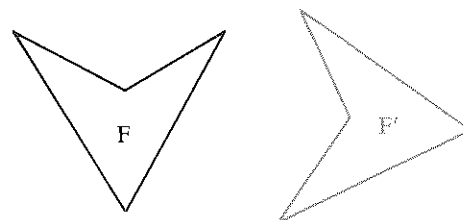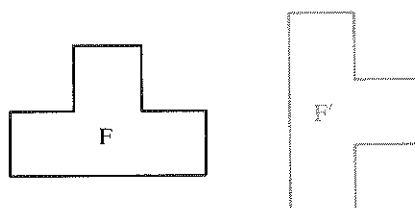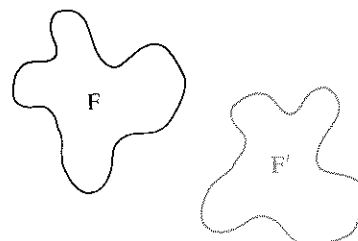
2. A triangle has vertices A(1,4), B(5,5), and O(0,0). Draw its image and find the coordinates of its vertices under a 180° rotation about O.

3. A quadrilateral has vertices A(1,3), B(4,7), C(6,4), and D(3,1). Draw its image and find the coordinates of its vertices under a 180° rotation about the origin.

4. A triangle has vertices P($-3,5$), Q($1,-7$), and R(2,1). Draw its image and find the coordinates of its vertices under a 90° rotation about the origin.

5. A quadrilateral has vertices K($-3,4$), L(1,6), M($6,-1$), and N($-4,-3$). Draw its image and find the coordinates of its vertices under a 90° rotation about the origin.

6. a) What can be said about the line segment joining any point P on a figure to its image P′, after a 180° rotation?
   b) How do the coordinates of a point and its image under a 180° rotation about (0,0) compare?

7. For each triangle shown below
   a) Copy the triangle.
   b) Draw its image under a 180° rotation about the rotation centre C.
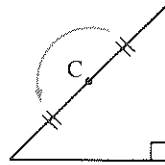   c) Name the figure formed by the triangle and its image.
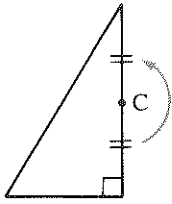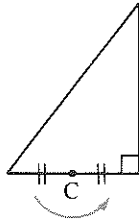
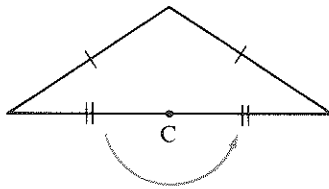   i)   ii)   iii)
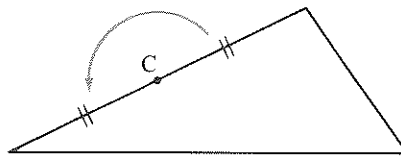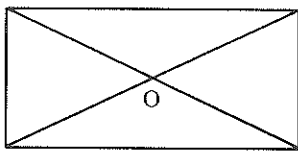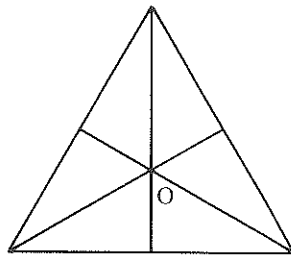
   iv)

        v)

8. Sketch the image of each figure under a 90°, a 180°, a 270°, and a 360° rotation about the rotation centre O. Which image maps onto the original figure when it is rotated through 180°?
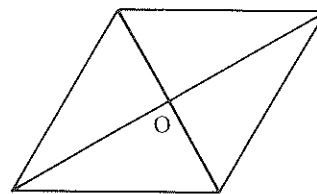   a)   b)   c)

9. What properties remain invariant under a rotation?

10. a) Graph this equation. $y = x + 3$
    b) Draw the image of the graph of the line in part a) under a 180° rotation about the origin.

11. a) Graph this equation. $y = 2x$
    b) Draw the image of the graph of the line in part a) under a 180° rotation about the origin.
    c) Explain the result.

PROBLEM SOLVING

Consider All Possibilities

A box is designed to hold a cube marked with the letters A through F as shown. In how many different ways can the cube be placed in the box?
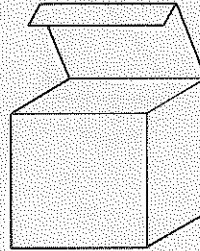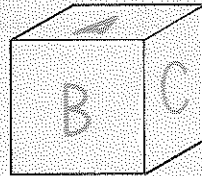
Understand the problem

- When are two ways of placing the cube in the box to be considered as "different"?
- Is there only one way to place the cube in the box so that face A is on top?
- What are we asked to find?

Think of a strategy

- Consider each letter in turn.
- Think of the number of ways the cube can be placed with that letter on top.

Carry out the strategy

- Count the number of ways of placing the cube in the box so that face A is on top.
- When A is on top, there are four different ways that A can appear, as shown in the diagram.
- Similarly, there are 4 different ways that the cube can be placed in the box when each of B, C, D, E, and F are on top.
- Therefore, the total number of ways of placing the cube in the box is 6 × 4, or 24.

Look back

- Try another method of counting the ways that the cube can be placed in the box. Do you get 24 by this method?

Solve each problem

1. How many lines of symmetry has a regular polygon of 12 sides (below left)?
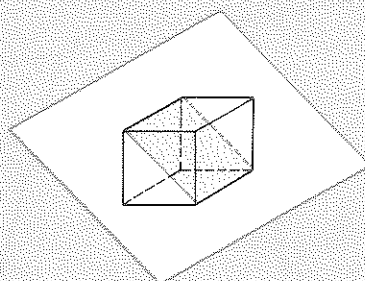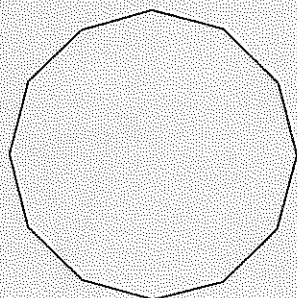
2. A plane which divides a solid into 2 parts such that each part is the reflection of the other in the plane is called a *plane of symmetry* of the solid (above right). How many planes of symmetry has a cube?

3. Place the integers from 1 to 6 inside the circles so that the sums along the sides of the triangle are equal (below left).
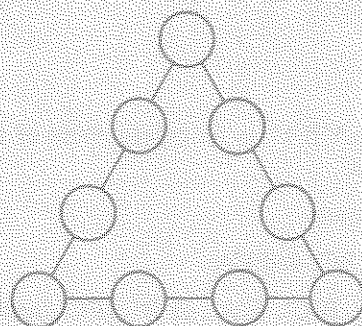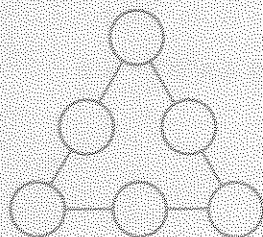
4. Place the integers from 1 to 9 inside the circles so that the sums along the sides of the triangle are equal (above right).

5. a) Choose any two prime numbers greater than 3. Find the difference of their squares. Is the difference a multiple of 24?

   b) Repeat part a) with other pairs of prime numbers.

   c) Make a conjecture about the difference of the squares of two prime numbers greater than 3.

   d) Test your conjecture using other pairs of prime numbers.

   e) Every prime number greater than 3 is either 1 more than a multiple of 4 (such as 17), or 3 more than a multiple of 4 (such as 19). Use this fact to prove your conjecture in part c).

INVESTIGATE

Enlargements and Reductions
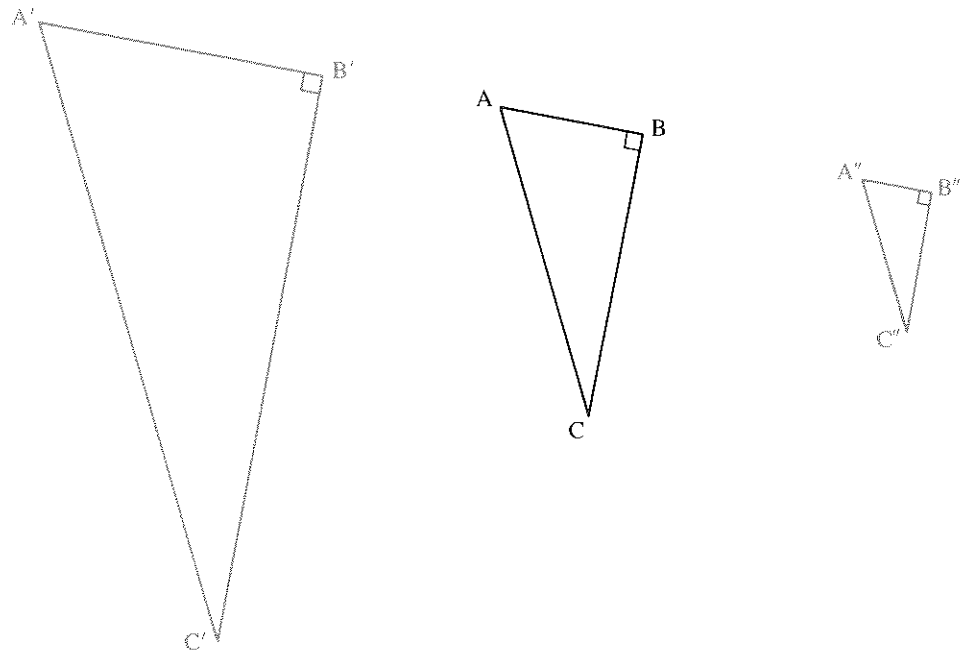
In the diagram, $\triangle A'B'C'$ is an enlargement of $\triangle ABC$ and $\triangle A''B''C''$ is a reduction of $\triangle ABC$.

1. Trace the diagram.

2. Measure AB, BC, and CA.

3. Measure $A'B'$, $B'C'$, $C'A'$, $A''B''$, $B''C''$, and $C''A''$.
   - What appears to be true about the lengths of line segments under an enlargement?
   - What appears to be true about the lengths of line segments under a reduction?

4. Measure $\angle ABC$, $\angle BCA$, and $\angle CAB$.

5. Measure $\angle A'B'C'$, $\angle B'C'A'$, $\angle C'A'B'$, $\angle A''B''C''$, $\angle B''C''A''$, and $\angle C''A''B''$.
   - What appears to be true about the measures of angles under an enlargement?
   - What appears to be true about the measures of angles under a reduction?

6. Find the areas of $\triangle ABC$, $\triangle A'B'C'$, and $\triangle A''B''C''$.
   - What appears to be true about the areas of figures under an enlargement?
   - What appears to be true about the areas of figures under a reduction?
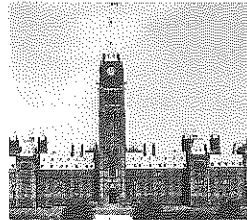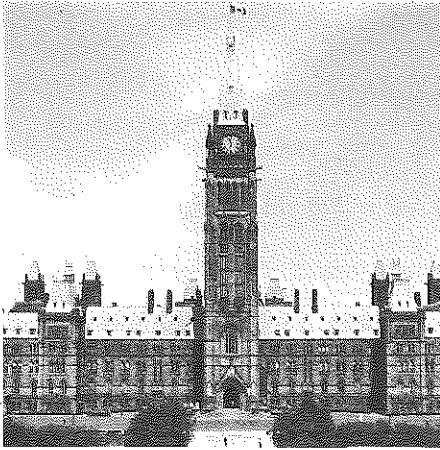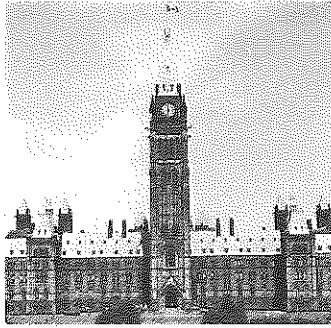
10-5 DILATATIONS

When a photograph has been increased in size, it has been *enlarged*. When a photograph is decreased in size, it has been *reduced*. In both cases, the shapes of the objects on the photographs are invariant, only their dimensions change.

Enlargements and reductions are examples of a transformation called a dilatation.

> A *dilatation* is a transformation that changes all dimensions by a factor k called the *scale factor*. For enlargements, k is greater than 1 and for reductions, k is between 0 and 1.

Example 1.   Here are two views of the Canadian flag; F′ is the dilatation image of F.
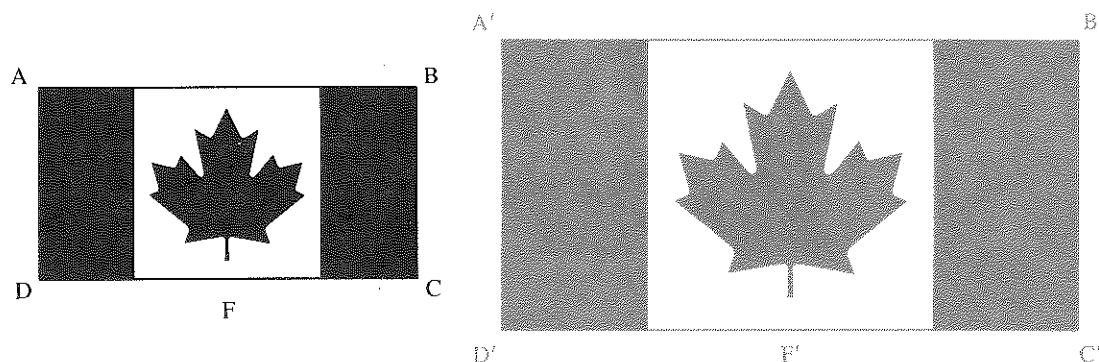
a) Find the scale factor k of the dilatation.
b) Compare the length-to-width ratios of the flags.
c) Find the areas of the flags.

Solution.   a) The scale factor k is the ratio of corresponding lengths.

$$k = \frac{A'B'}{AB} \left(\text{or } \frac{B'C'}{BC} \text{ or } \frac{C'D'}{CD} \text{ or } \frac{A'D'}{AD} \right)$$

By measuring, $A'B' = 7.5$ cm and $AB = 5.0$ cm

$$k = \frac{7.5}{5.0}$$

$$= 1.5$$

The scale factor k of the dilatation is 1.5.

b) For flag F, $\dfrac{\text{length}}{\text{width}} = \dfrac{5.0}{2.5}$

$$= 2.0$$

For flag F′, $\dfrac{\text{length}}{\text{width}} = \dfrac{7.5}{3.8}$

$$\doteq 2.0$$

The length-to-width ratio of the flag is unchanged by the dilatation.

c) For flag F, area $= (5.0 \times 2.5)$ cm²
$$= 12.5 \text{ cm}^2$$
For flag F′, area $= (7.5 \times 3.8)$ cm²
$$= 28.5 \text{ cm}^2$$

In *Example 1*, the area of F′ is *not* 1.5 times the area of F. Since each dimension of F′ is 1.5 times the corresponding dimension of F, the area of F′ is $(1.5)^2$, or 2.25 times the area of F.

Use your calculator to show that $\dfrac{28.5}{12.5} \doteq 2.25$

In *Example 1*, if we join corresponding points on the flags and extend the lines, we find that they meet at a point called the *dilatation centre*.
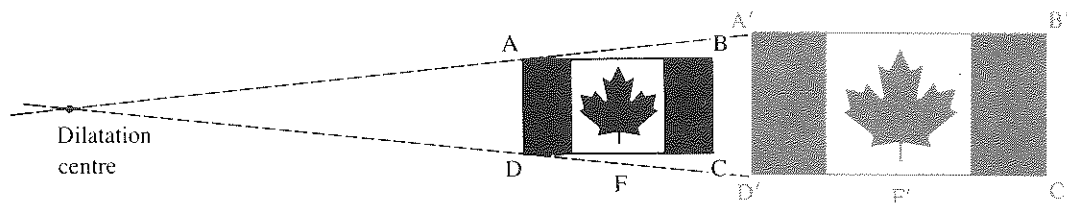
Here are some properties of dilatations. You may have discovered some of these in the *INVESTIGATE* preceding this section.

Under a dilatation with scale factor k

- The length of an image line segment is k times the length of the original line segment.
- The measures of angles are invariant.
- The area of an image figure is k^2 times the area of the original figure.
- Lines through corresponding points meet at the dilatation centre.

Example 2. The map is a dilatation of the city of Lethbridge with a scale of 1 : 112 000. That is, x centimetres on the map corresponds to an actual distance of 112 000x centimetres. Use the map to determine the actual distance between 43 Street and 13 Street along 5 Avenue.

Solution. By measuring, the map distance between 43 Street and 13 Street along 5 Avenue is 2.9 cm.

Actual distance is 112 000 × 2.9 cm
$$= 324\ 800 \text{ cm}$$
$$= 3.248 \text{ km}$$

It is approximately 3.2 km from 43 Street to 13 Street along 5 Avenue.

LETHBRIDGE, ALBERTA
SCALE: 1: 112000
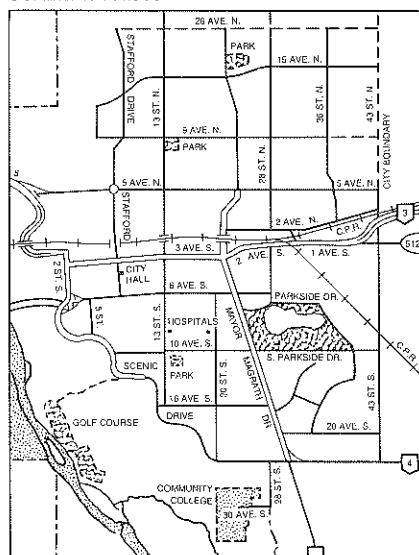
Fuck

Example 3. A dilatation maps figure F onto figure F'.
a) Find the scale factor.
b) Locate the dilatation centre.
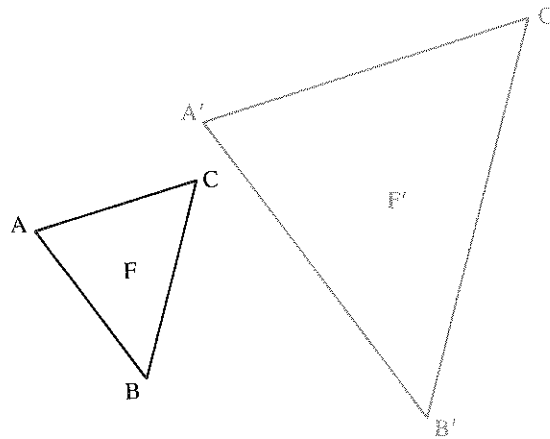
Solution. a) Trace the diagram.
Measure the corresponding lengths AB and A'B'.

The scale factor k of the dilatation is the ratio $\dfrac{A'B'}{AB}$.

$$k = \frac{A'B'}{AB}$$
$$= \frac{5.0}{2.5}$$
$$= 2$$

The scale factor is 2.

b) Join A' to A and extend A'A.
Join B' to B and extend B'B.
The intersection of A'A and B'B is the dilatation centre D.

EXERCISES 10-5

Ⓐ
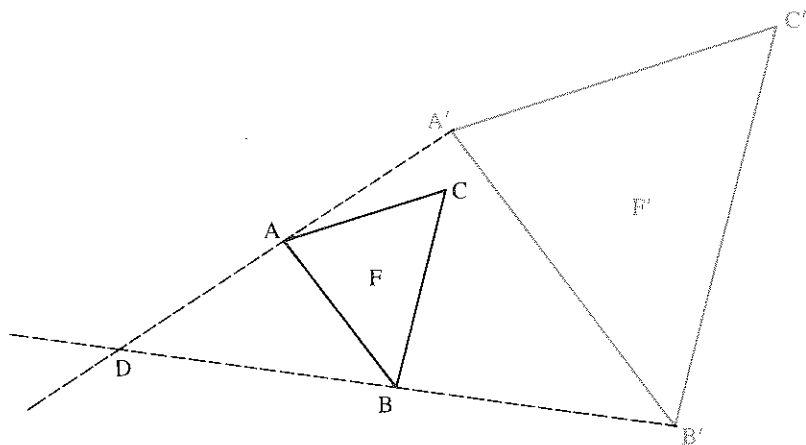
1. A figure undergoes a dilatation with the given scale factor. Is the image an enlargement or a reduction?

a) 3   b) 7   c) $\frac{1}{2}$ d) $\frac{3}{2}$ e) $\frac{1}{4}$ f) $\frac{5}{6}$

2. Each diagram shows a figure F and its image F′ under a dilatation. Trace each diagram. Locate the dilatation centre and determine the scale factor.

a)

b)

c)

d)
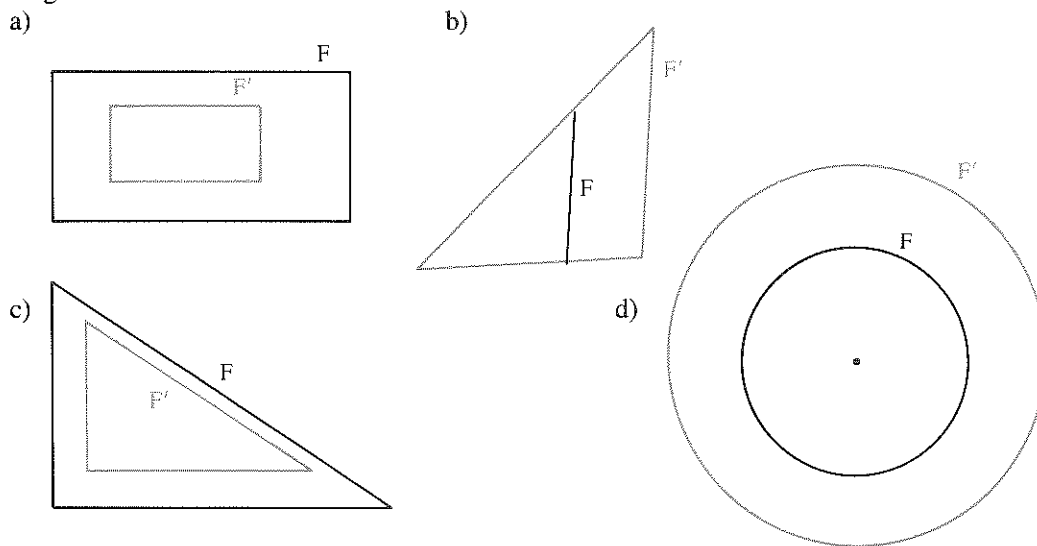
3. The larger photograph is a dilatation image of the smaller photograph.

Measure the photographs.
a) Find the scale factor of the dilatation.
b) Find the length-to-width ratios of the photograph and its enlargement.
c) Find the areas of the photograph and its enlargement.

4. Trace each figure and its dilatation image.
    i) Find the dilatation centre.
    ii) Determine the scale factor of the dilatation.
    iii) Compare the areas of the figure and its image.
    iv) Compare the angles of the figure and its image.

a)
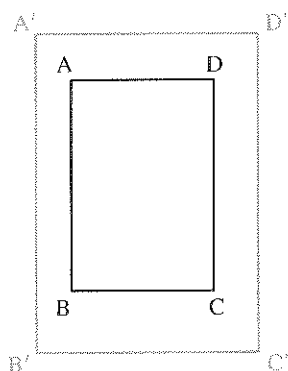
b)
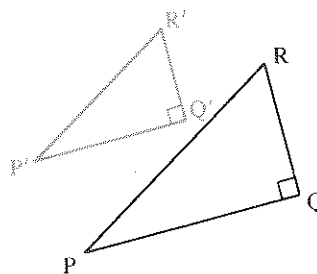
c)
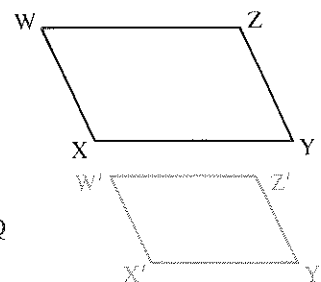
5. A′B′C′D′E′ is a dilatation image of pentagon ABCDE. Trace the diagram onto squared paper.
    a) Find the dilatation centre.
    b) Compare the lengths of corresponding sides.
    c) What is the scale factor of the dilatation?
    d) Compare the measures of corresponding angles.
    e) Compare the areas of the pentagon and its image.
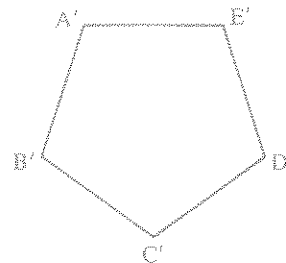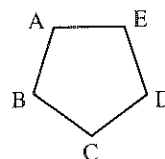
6. Triangle ABC has sides of length 5 cm, 12 cm, and 13 cm. Find the lengths of the sides of its dilatation image for each scale factor.

    a) 2
    b) 5
    c) $\dfrac{1}{2}$
    d) $\dfrac{3}{4}$

7. Using O as the dilatation centre, draw the image of each triangle with each scale factor.
    i) 2.5   ii) 0.75

a)
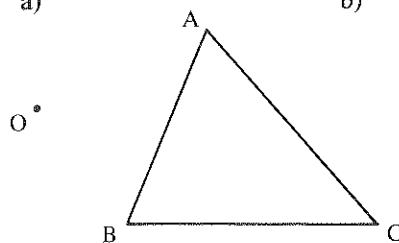
b)
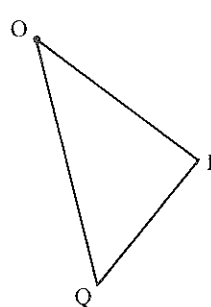
c)
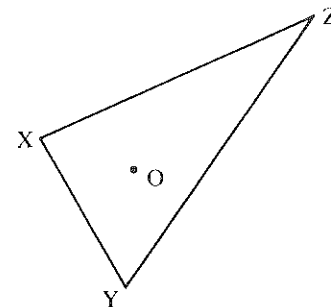
8. On a blueprint, the dimensions of a building are 640 mm by 360 mm. Determine the scale factor of the dilatation if the actual building has these dimensions.
   a) 64 m by 36 m
   b) 96 m by 54 m
   c) 48 m by 27 m
   d) 80 m by 45 m

9. From the map in *Example 2*, find the actual distance between each pair of locations.
   a) 9 Ave. N. and 16 Ave. S. along 13 Street
   b) 13 Street and 28 Street along 5 Ave.
   c) the two points where the CPR tracks cross 43 Street

10. The scale for this map of Niagara Falls is 1 : 91 000. How far is it between each pair of locations?
    a) Dorchester Rd. to River Rd. along Morrison St.
    b) Thorold Stone Rd. to McLeod Rd. along Drummond Rd.
    c) the corner of Dorchester Rd. and Dunn St. to the corner of Bridge St. and Stanley Ave.

NIAGARA FALLS
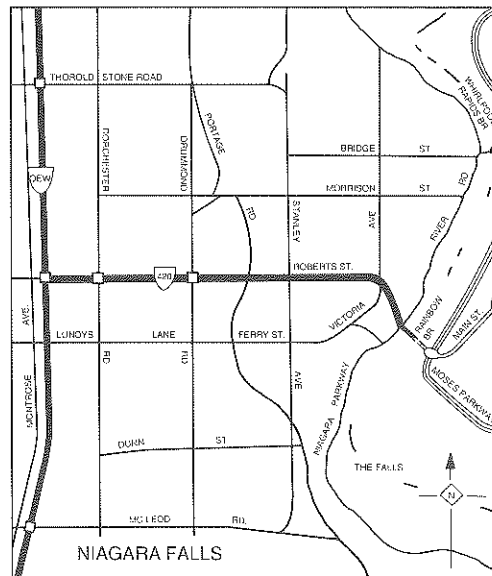
11. O is the dilatation centre for $\triangle ABC$ and its image $\triangle A'B'C'$.
    a) Determine the scale factor k.
    b) Find these ratios.
       $$\frac{OA'}{OA}, \quad \frac{OB'}{OB}, \quad \frac{OC'}{OC}$$
    c) Copy the diagram and draw $\triangle A''B''C''$, the dilatation image of $\triangle ABC$ with scale factor 2.
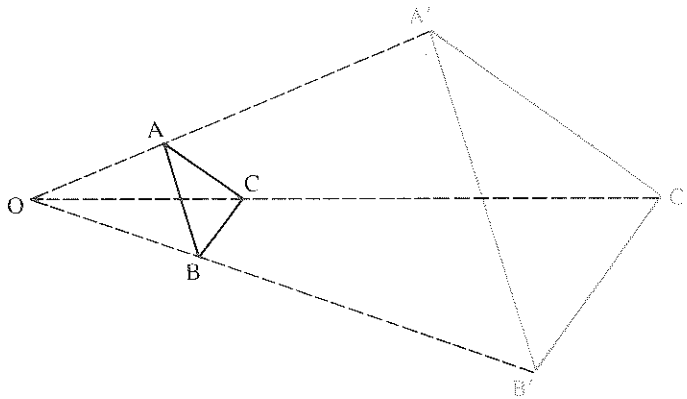
10-6 SIMILAR FIGURES

Two surveyors need to know the distance AB across a pond, which cannot be measured directly.

The surveyors located points D and E such that DE is parallel to AB, and DE can be measured. They completed the triangle at point C.

Triangle CAB can be considered as a dilatation image of $\triangle CDE$ about the dilatation centre C.

Hence, corresponding sides of the triangles are in the same ratio, which is the scale factor of the dilatation.

The surveyors know that $\dfrac{AB}{DE} = \dfrac{AC}{DC}$.

They measured DE, AC, and DC, and these lengths are labelled on the diagram.

Hence, $\dfrac{AB}{25.7} = \dfrac{85.5}{63.0}$

$$AB = \frac{85.5}{63.0}(25.7)$$

$$\doteq 34.9$$

The distance AB across the pond is about 35 m.
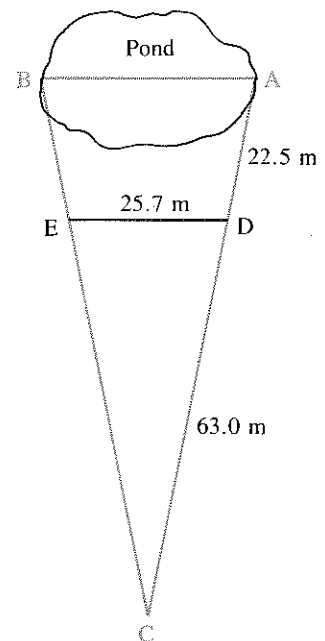
If one geometric figure is the image of another under a dilatation, then the figures are said to be similar.
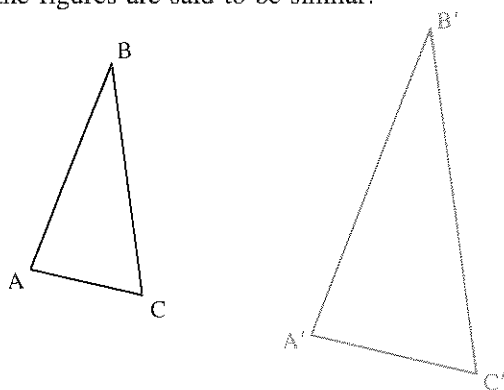
Two triangles ABC and A′B′C′ are *similar* if the ratios of corresponding sides are equal.

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

Two triangles ABC and A′B′C′ are *similar* if pairs of corresponding angles are equal.

$$\angle A = \angle A'; \angle B = \angle B'; \angle C = \angle C'$$

To indicate that △ABC is similar to △A′B′C′, we write △ABC ∼ △A′B′C′.

Example 1. Given the figure with the indicated sides parallel

a) Explain why △PQT is similar to △PRS.

b) Explain why $\dfrac{QT}{RS} = \dfrac{PT}{PS}$.
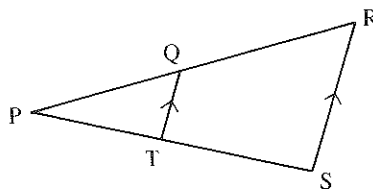
Solution. a) ∠PTQ and ∠PSR are corresponding angles.
∠PQT and ∠PRS are corresponding angles.
Since QT is parallel to RS,
∠PTQ = ∠PSR
∠PQT = ∠PRS
∠P is common to △PQT and △PRS.
Since pairs of corresponding angles are equal, △PQT ∼ △PRS.

b) The ratios of corresponding sides of △PQT and △PRS are equal.
Therefore, $\dfrac{QT}{RS} = \dfrac{PT}{PS}$

Example 2.   Find the values of x and y.
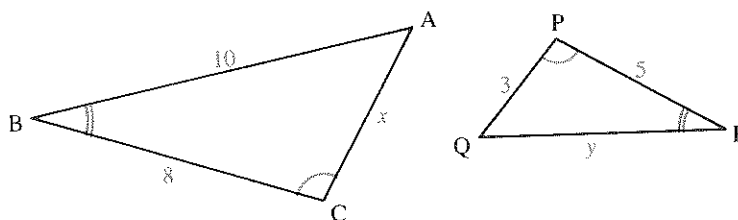
Solution.   In $\triangle ABC$ and $\triangle PQR$

$\angle B = \angle R$

$\angle C = \angle P$

Since two pairs of angles in the triangles are equal, the third pair of angles must be equal.

$\angle A = \angle Q$

Since pairs of corresponding angles are equal, $\triangle ABC$ is similar to $\triangle QRP$. Hence, the ratios of corresponding sides of $\triangle ABC$ and $\triangle QRP$ are equal.

$$\frac{AB}{QR} = \frac{BC}{RP} = \frac{AC}{QP}$$

Substitute the given values.

$$\frac{10}{y} = \frac{8}{5} = \frac{x}{3}$$

Consider these expressions two at a time.

$$\frac{10}{y} = \frac{8}{5} \qquad\qquad\qquad \frac{8}{5} = \frac{x}{3}$$

$$5(10) = 8y \qquad\qquad\qquad \frac{8(3)}{5} = x$$

$$y = \frac{50}{8}$$

$$= 6.25 \qquad\qquad\qquad x = \frac{24}{5}$$

$$= 4.8$$

Example 3.   The largest international standard paper size is designated by the symbol A0. The A0 size is a rectangle of length 1.19 m and width 0.84 m. The next largest size, A1, is a rectangle similar to A0 but with half the area. What are the dimensions of A1-size paper?
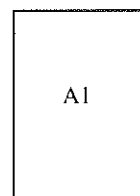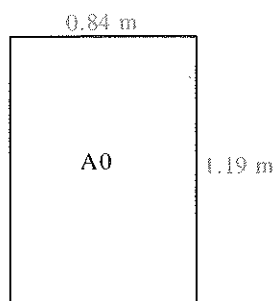
Solution. Since the two rectangles are similar, rectangle A1 is the image of rectangle A0 under a dilatation with scale factor k.

Since the area of rectangle A1 is one-half that of rectangle A0,

$$k^2 = \frac{1}{2}$$

$$k = \frac{1}{\sqrt{2}}$$

Therefore, $\dfrac{\text{length of rectangle A1}}{\text{length of rectangle A0}} = \dfrac{1}{\sqrt{2}}$

$$\text{length of rectangle A1} = \frac{1}{\sqrt{2}}(1.19)$$

$$\doteq 0.84$$

and $\dfrac{\text{width of rectangle A1}}{\text{width of rectangle A0}} = \dfrac{1}{\sqrt{2}}$

$$\text{width of rectangle A1} = \frac{1}{\sqrt{2}}(0.84)$$

$$\doteq 0.59$$

The dimensions of A1-size paper are 0.84 m by 0.59 m.

EXERCISES 10-6

Ⓐ

1. State which figures are similar.

Ⓑ
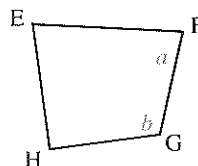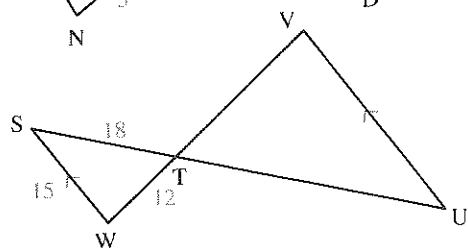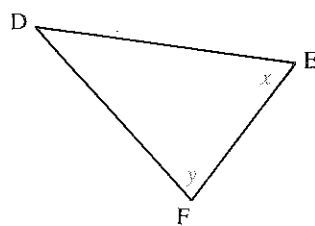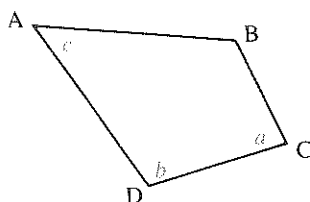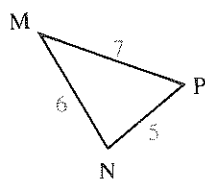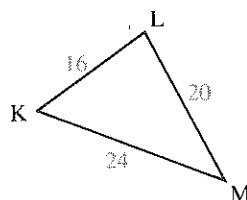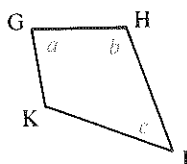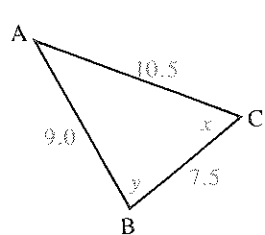
2. Show that:

a) $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$

b) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$.
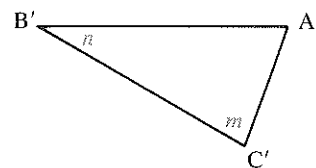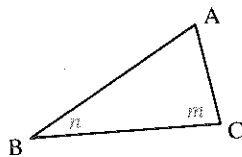
3. Show that:

a) $\dfrac{DE}{DG} = \dfrac{DF}{DH}$

b) $\dfrac{DE}{DG} = \dfrac{EF}{GH}$.
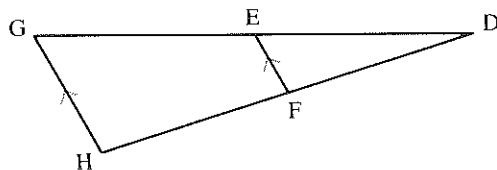
4. Show that:

a) $\dfrac{TR}{TV} = \dfrac{TS}{TU}$

b) $\dfrac{TR}{TV} = \dfrac{RS}{VU}$.
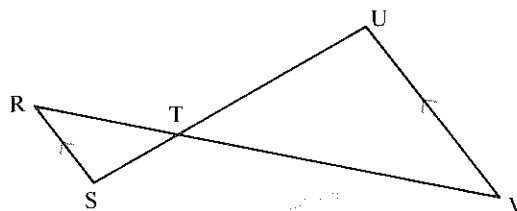
5. Find the values of x and y.
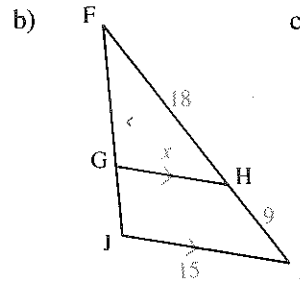
a)

b)

c)

d)

e)

f)
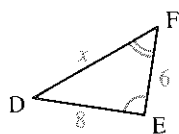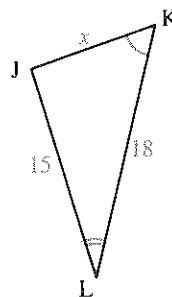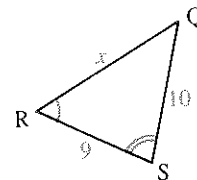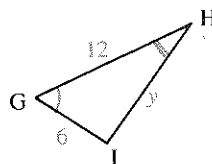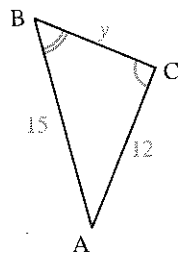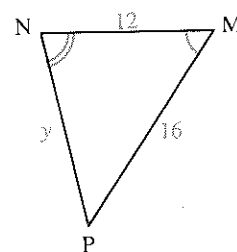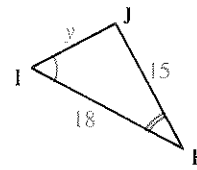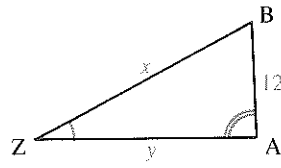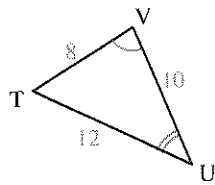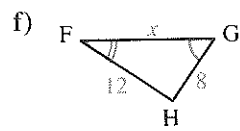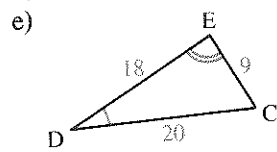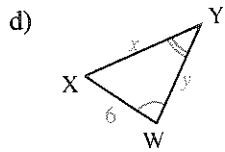
6. Find the values of x.

a)

b)

c)

d)

e)

f)
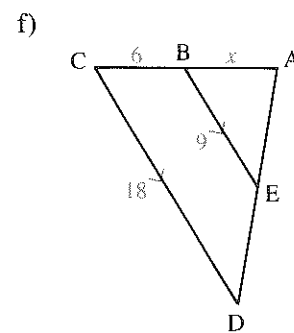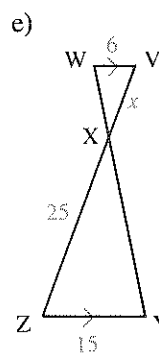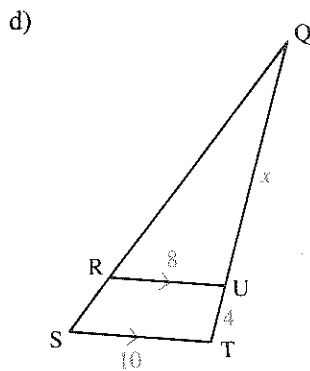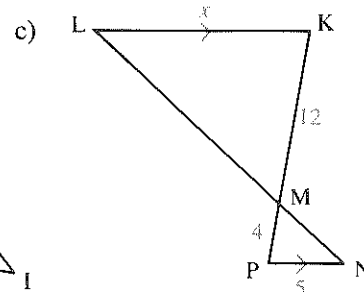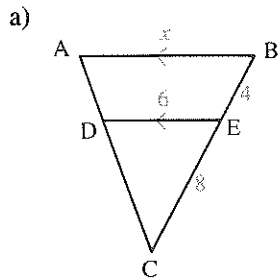
7. To find the distance PQ across a farm pond, Marty marks out points R and S so that RS is parallel to PQ. By measuring, she finds that RS = 5.7 m, OP = 19.5 m, and OS = 4.2 m. What is the distance PQ?

Pond
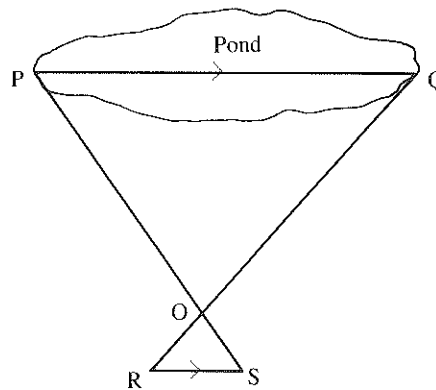
8. The shadow of a telephone relay tower is 32 m long on level ground. At the same time, a boy 1.8 m tall casts a shadow 1.5 m long. What is the height of the tower?

9. Karen is 37.5 m from a church. She finds that a pencil, 4.8 cm long, which is held with its base 60 mm from her eye, just blocks the church from her sight. How high is the church?
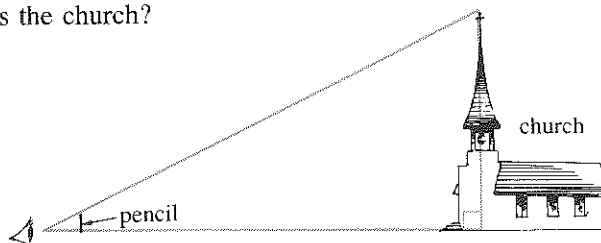

10. The shape of A2-size paper is a rectangle similar to the A1 size but with half its area.
    a) Use the data in *Example 3* to find the dimensions of the A2 size.
    b) What symbol designates paper measuring 30 cm by 21 cm?

11. Sunil has a photograph measuring 20 cm by 25 cm. He wishes to get a copy that is three-quarters the area but with the same length-to-width ratio. What will be the dimensions of the copy?

12. For the photograph in *Exercise 11*, what would be the dimensions of a copy with double the area of the original but with the same shape?
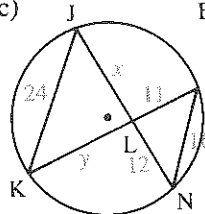
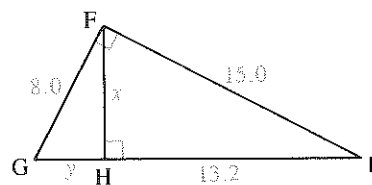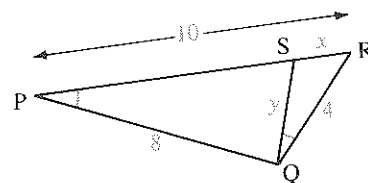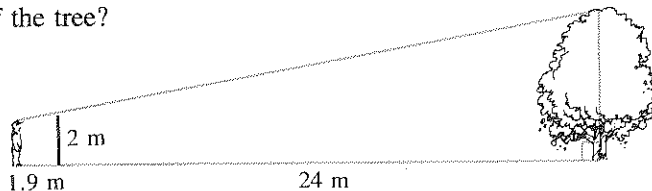13. Find the values of *x* and *y*.

    a)

    b)

    c)
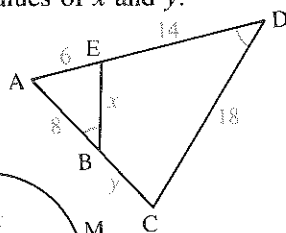
    d)

14. To determine the height of a tree, Jerry places a 2 m rod 24 m from the tree. He finds that he can just align the top of the rod with the top of the tree when he stands 1.9 m from the rod. If Jerry's eyes are 1.6 m from the ground, what is the height of the tree?

10-7 APPLICATIONS OF TRANSFORMATIONS

We can solve certain apparently difficult problems using transformations.
Consider the problem posed at the beginning of this chapter.

Example 1. A pumping station is to be built somewhere along a pipeline to serve tanks at two points near the pipeline. Where should the pumping station be located so that the total length of the pipe from the pipeline to the tanks is a minimum?

Solution. Draw a diagram. Let A and B represent the tanks, and let l represent the pipeline.
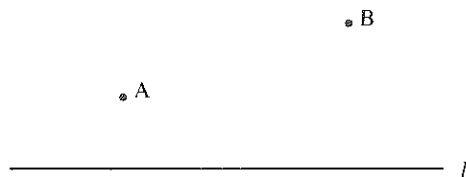
Locate B', the reflection image of B in l. The shortest distance between A and B' is the line segment joining them. Let AB' intersect l at P. Join PB. PB is the reflection image of PB' in l.
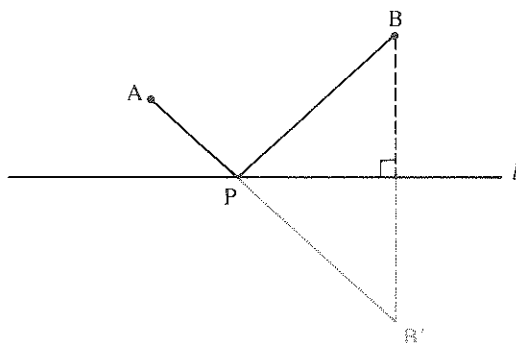
Since AP + PB' is the shortest distance from A to B', AP + PB is the shortest distance from A to l to B.
The pumping station should be built at P.

When using transformations to solve problems, we make use of the fact that reflections, rotations, and translations preserve lengths, angles, and areas. *Example 1* involved the invariance of length under a reflection.

Although dilatations do not preserve length, they do preserve the ratios of lengths. This property is used in the following example.

Example 2.   The image on the screen of a drive-in theatre is projected from a film 35 mm wide. The film is 150 mm in front of the light source and the screen is 14 m high. How far is the screen from the light source?

Solution.   Draw a diagram.
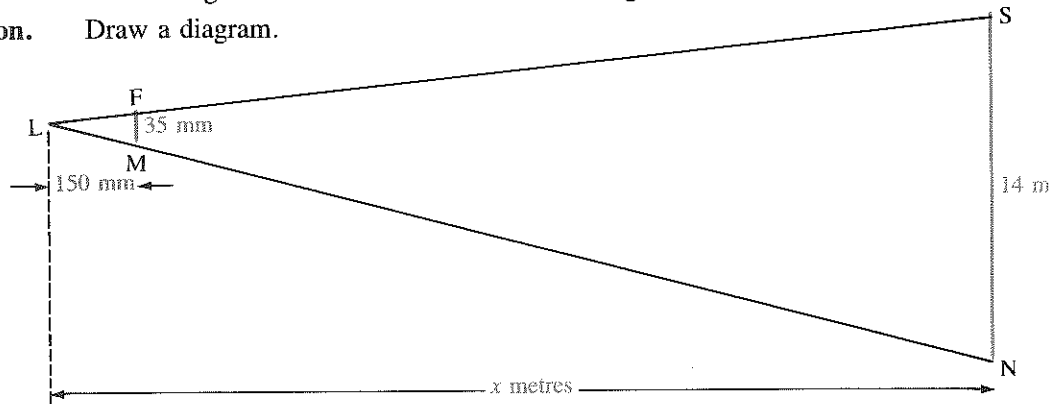
Let the distance from the light source L to the screen SN be x metres. Since the film FM maps onto the screen SN, $\triangle$LFM maps onto $\triangle$LSN under a dilatation with centre L.

Hence, $\triangle$LFM is similar to $\triangle$LSN.

Therefore, the ratios of corresponding lengths in the two triangles are equal.

$$\frac{150}{FM} = \frac{x}{SN}$$

$$\frac{150}{35} = \frac{x}{14}$$

$$x = \frac{150 \times 14}{35}$$

$$= 60$$

The screen is 60 m from the light source.

EXERCISES 10-7

Ⓐ

1. Show how to locate point R on line l such that PR + QR is a minimum.

───────────────── l

° P

° Q

2. a) Plot the points A(3,5) and B(7,2) on a grid.
   b) Locate the point M on the x-axis such that AM + MB is a minimum.
   c) Locate the point N on the y-axis such that AN + NB is a minimum.

3. Using points A and B of *Exercise 2*, locate point M on the y-axis and point N on the x-axis such that AM + MN + NB is a minimum.

4. Copy this diagram.

a) Draw the shortest path from P to l_1 to l_2 to Q.
b) Draw the shortest path from P to l_2 to l_1 to Q.
c) In finding the shortest path from P to Q, does it matter which line (l_1 or l_2) you go to first?

5. Two adjacent cushions of a billiard table can be represented by the x- and y-axes on a grid. Let the points B(3,6) and W(12,6) represent the positions of the brown ball and a white ball respectively.
a) Find the coordinates of a point on the x-axis at which the white ball should be aimed so as to rebound and hit the brown ball.
b) i) Find the coordinates of the point on the x-axis at which the white ball should be aimed so as to rebound from the x-axis to the y-axis and hit the brown ball.
    ii) What are the coordinates of the point where the ball hits the y-axis?

6. A fly lands at the point A(4,6) on a grid. It walks to the y-axis, then to the x-axis, and finally stops at the point B(8,3). What is the shortest distance the fly could have walked?

7. A film negative, measuring 13 mm by 17 mm, is 32 mm from a camera lens. What are the maximum dimensions of an object that can be photographed from a distance of 12.4 m?

8. To determine the distance AB across a river, a surveyor places a marker at C. She finds a point A′ in alignment with C and a prominent feature A on the far side of the river. Finally, she locates positions B and B′ so that $\angle ABC = \angle A'B'C = 90°$. If A′B′ = 3.8 m, B′C = 5.2 m, and BC = 14.7 m, how wide is the river at AB?
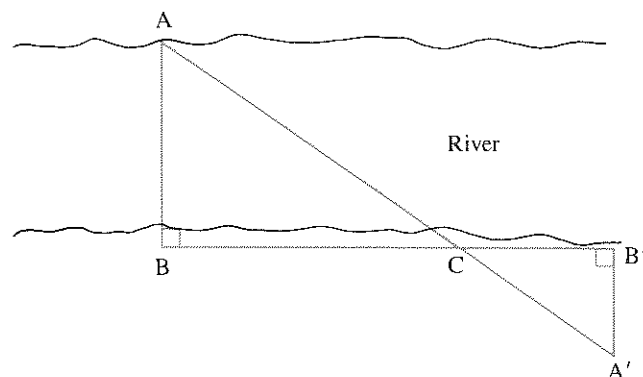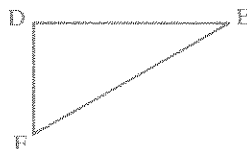
Review Exercises

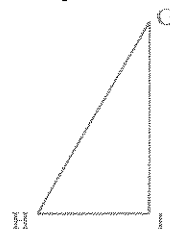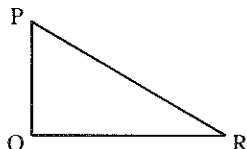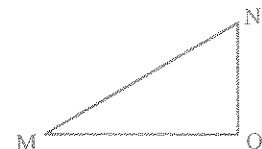1. Each triangle is an image of △PQR under a transformation. Identify each transformation.
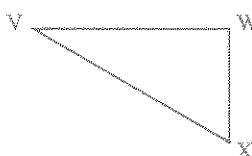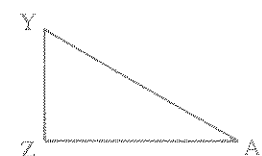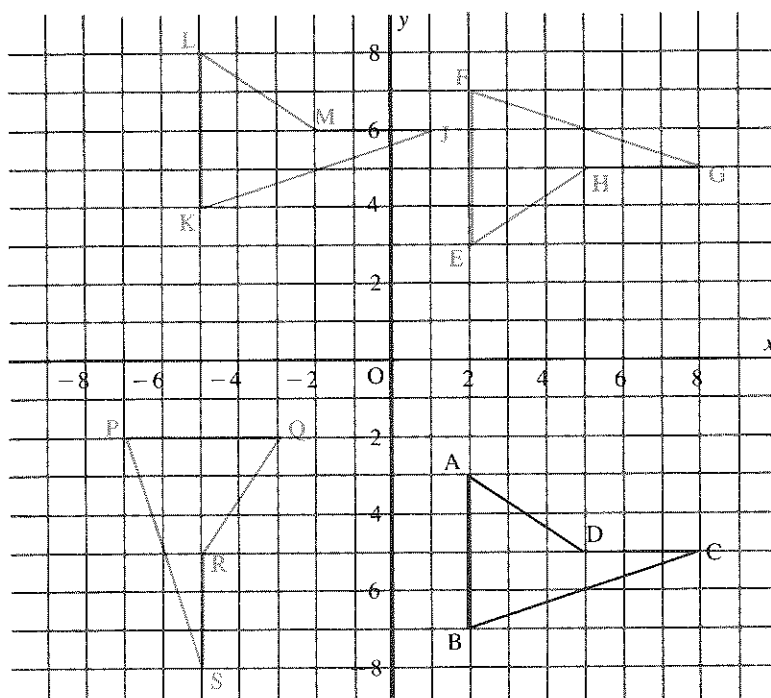
a)
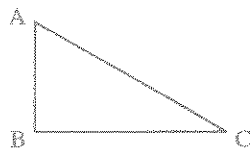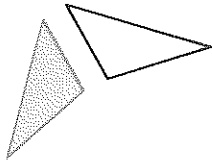
b)

c)

d)
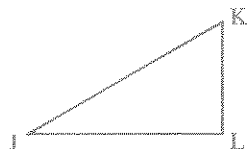
e)

f)
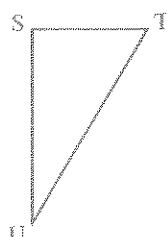
g)

h)

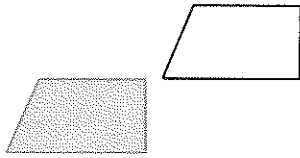2. Each quadrilateral is an image of quadrilateral ABCD under a transformation. Identify each transformation.

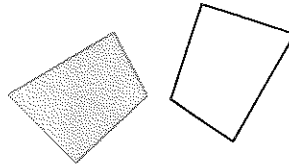3. State the type of transformation that appears to map each polygon onto its colored image.
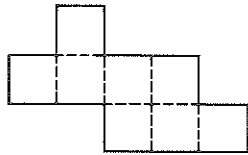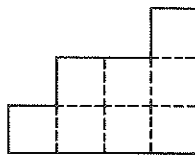
a)

b)

c)

4. Copy each diagram. Divide it along the broken lines into two congruent parts. Name the transformation needed to map one part onto the other.
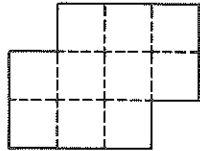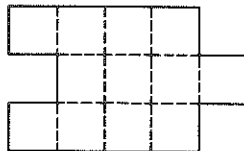
a)

b)

c)

d)

5. A translation maps $(-3,-1)$ onto $(2,-3)$.
   a) Find the images of $A(-2,4)$, $B(1,-5)$, and $C(4,1)$ under this translation.
   b) If $P'(2,-3)$, $Q'(0,4)$, and $R'(-5,1)$ are image points under this translation, find the original points.

6. Graph the points $A(2,5)$, $B(-3,1)$, and $C(1,-4)$. Find the image of each point under a reflection in:
   a) the x-axis
   b) the y-axis.

7. Graph the points $L(-2,-3)$, $M(4,-3)$, and $N(4,5)$. Find the image of each point under a $180°$ rotation about the origin.

8. The reflection images of $A(1,4)$, $B(3,-2)$, and $C(5,1)$ are $A'(-3,4)$, $B'(-5,-2)$, and $C'(-7,1)$. Graph both triangles and draw the reflection line.

9. Copy $\triangle XYZ$ and draw its dilatation image (about a dilatation centre of your choice) using each scale factor.
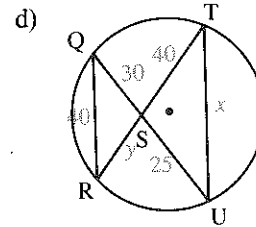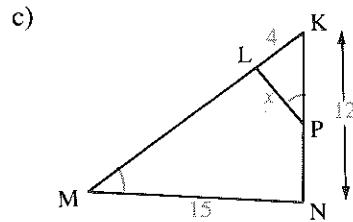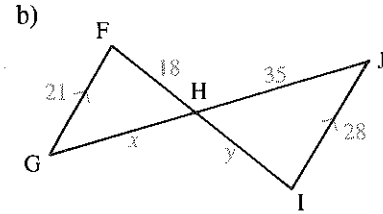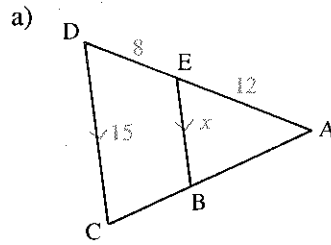
   a) 2
   b) $\frac{3}{4}$

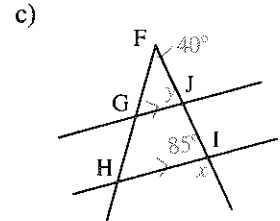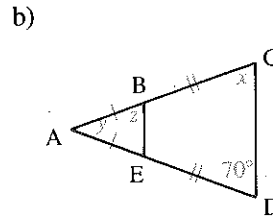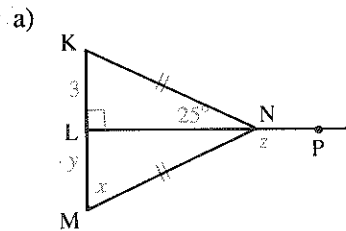   Compare the area of each image with the area of $\triangle XYZ$.
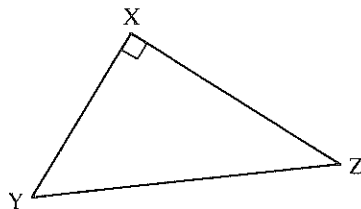
10. Find the values of x and y.
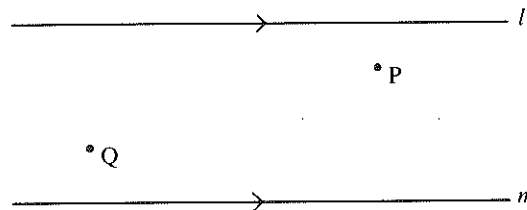
a)

b)

c)

d)

11. A building 18.2 m tall casts a shadow 7.1 m long. At the same time, how long is the shadow of a building that is 15.7 m tall?

12. A poster measures 30 cm by 45 cm. A copy is made, which has three times the area. What are its dimensions?

13. Find the values of x, y, and z.

a)

b)

c)

14. Copy the diagram. Draw the shortest path from P to l to m to Q.

15. Copy this diagram. Translate the exterior angles at vertices B, C, D, and E to vertex A. What is the sum of the exterior angles of the polygon?